Admissible Symmetries of the Electromagnetic Field in LRS Spacetimes

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Outline

- Locally Rotationally Symmetric spacetimes
- Physical model, solution classes

- How did we get here? Applying constraints

- EM field symmetries: earlier assumptions
- EM field, more generally

- Related work
- Conclusions

Locally Rotationally Symmetric Spacetimes

- Spacetimes with perfect fluid flow vector u^i studied by Ellis (1967) and Stewart and Ellis (1968), whose notation we follow.
- *A spacetime is locally LRS in a neighborhood* $N(P_0)$ *of a point* P_0 *if at each point* $P \in N(P_0)$ *there exists a non-discrete subgroup* g *of the Lorentz group in the tangent space* T_P *which leaves invariant* u^i*, the curvature tensor, and their derivatives up to* 3^{rd} *order.*
- g operates in a subspace of T_P orthogonal to u^i and so is a 1- or 3-dimensional group of rotations in T_P.
- If g is 3-dimensional, then have a RW model.
  These are included in the solutions for g 1-dimensional.
- Cosmological models with limited symmetry groups.

Physical Model

- **Perfect fluid** with flow u^i. $u^i u_i = -1$. $h_{ij} = g_{ij} + u_i u_j$.

Expansion	$\theta = u^i{}_{;i}$.	
Acceleration	$\dot{u}_i = u_{i;j} u^j$.	$\dot{u} = \sqrt{\dot{u}_i \dot{u}^i}$.
Vorticity	$\omega_{ij} = u_{[i;j]} + \dot{u}_{[i} u_{j]}$.	$\omega = \sqrt{\frac{1}{2}\omega_{ij}\omega^{ij}}$.
Shear	$\sigma_{ij} = u_{(i;j)} + \dot{u}_{(i} u_{j)} - \frac{1}{3}\theta h_{ij}$.	$\sigma = \sqrt{\frac{1}{2}\sigma_{ij}\sigma^{ij}}$.

- **Viscous** with, in rest frame of u_i,
  energy density μ, scalar pressure p,
  energy flux q_i, anisotropic pressure π_{ij}.

- **Non-interacting EM field**

$$F_{ij} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}.$$

Charge density ϵ.

Physical Model

- Energy-momentum tensor

$$T_{ij} = \mu u_i u_j + p h_{ij} + 2u_{(i}q_{j)} + \pi_{ij} + \tau_{ij},$$

where

$$\tau_{ij} = \frac{1}{4} g_{ij}(F_{kl}F^{kl}) - F_{ik}F_j{}^k.$$

Tetrad choice

- At each point introduce a local orthonomal frame $\{\mathbf{e}_a\}$. Convention: coord ix $i, j, ...$, frame ix $a, b, ...$, geom ob $\mathbf{bold}$.

$$\mathbf{K} = K^i{}_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j = K^a{}_b \mathbf{e}_a \otimes \mathbf{e}^b$$

.

- Choose $\mathbf{e}_0$ to lie along u^a and $\mathbf{e}_1$ along an axis of symmetry.
- LRS implies all cov. defined spacelike vectors are parallel to $\mathbf{e}_1$ and spacelike parts of cov. defined rank 2 tensors have diagonal form with (22) and (33) components equal.
- Write

$$\pi_{ab} = \mathrm{diag}(0, 2\pi, -\pi, -\pi)$$

$$\sigma_{ab} + \frac{1}{3}h_{ab} = \mathrm{diag}(0, \alpha, \beta, \beta)$$

Tetrad choice

- Stewart and Ellis, Thm 1: Can choose tetrad such that

$$
\begin{aligned}
[\mathbf{e}_0, \mathbf{e}_1] &= \dot{u}\mathbf{e}_0 - \alpha\mathbf{e}_1 \\
[\mathbf{e}_0, \mathbf{e}_2] &= -\beta\mathbf{e}_2 \\
[\mathbf{e}_0, \mathbf{e}_3] &= -\beta\mathbf{e}_3 \\
[\mathbf{e}_2, \mathbf{e}_3] &= -2\omega\mathbf{e}_0 - k\mathbf{e}_1 + s\mathbf{e}_3 \\
[\mathbf{e}_3, \mathbf{e}_1] &= -a\mathbf{e}_3 \\
[\mathbf{e}_1, \mathbf{e}_2] &= a\mathbf{e}_2,
\end{aligned}
$$

  with $\partial_3 s = 0$.

- $\mu, p, q, \pi, \tau, \omega, \dot{u}, \alpha, \beta, a, k$ are cov. defined so have $\partial_2 = \partial_3 = 0$.

  Let $r = \partial_2 s - s^2$.

LRS Classes

- Stewart and Ellis, Thm 2: Have 3 disjoint and exhaustive cases

  Class I: $\omega \neq 0, k = 0.$ $\alpha = \beta = 0.$
  Class II: $\omega = 0, k = 0.$
  Class III: $\omega = 0, k \neq 0.$ $\dot{u} = a = \partial_1 k = \partial_1 \alpha = \partial_1 \beta = 0.$

- Isometry groups:

Class	**Group**	**Orbit**	**Specialization**
Ia	G_5	spacetime	$a = \dot{u} = 0$
Ib	G_4	$\{x^1 = C^1\}$	$a = 0, \dot{u} \neq 0$
Ic	G_4	$\{x^1 = C^1\}$	$a \neq 0, \dot{u} = 0$
Id	G_4	$\{x^1 = C^1\}$	$a \neq 0, \dot{u} \neq 0$
IIa	G_3	$\{x^0 = C^0, x^1 = C^1\}$	$a = 0$
IIb	G_4	$\{x^1 = C^1\}$	$a \neq 0, \dot{u} = 0$
IIc	G_3	$\{x^0 = C^0, x^1 = C^1\}$	$a \neq 0, \dot{u} \neq 0$
IIIa	G_5	spacetime	$\beta = 0$
IIIb	G_4	$\{x^0 = C^0\}$	$\beta \neq 0$

- Class IIc most interesting.

Basic Equations for Class II

- Jacobi identities:

$$\partial_1 s = as$$
$$\partial_1 \beta + \partial_0 a = -\beta \dot{u} - \alpha a$$
$$\partial_0 s = -\beta s$$

- Einstein field equations:

$$\partial_0 \alpha = -\frac{1}{2}(\mu + p) + \beta^2 - \alpha^2 - a^2 + r + \partial_1 \dot{u} + \dot{u}^2$$
$$\partial_0 \beta = \frac{1}{2}\Big(\Lambda - p - 3\beta^2 + a^2 - r - 2a\dot{u} - \tau\Big)$$
$$\partial_1 a = \frac{1}{2}\Big(\Lambda + \mu - \beta^2 - 2a\beta + 3a^2 - r + \tau\Big)$$
$$\partial_1 \beta = a(\beta - \alpha) - \frac{1}{2}\tau$$

- Bianchi identities:

$$\partial_0 \mu = -(\mu + p)(\alpha + 2\beta)$$
$$\partial_1 p = -(\mu + p)\dot{u}$$

How Did We Get Here?

- Want to explore solutions to the field equations satisfying certain properties.
- Adding an isometry to LRS over simplifies.
- Want to enforce, e.g. equation of state of certain form, intrinsic symmetries on sub-manifold families.

- Add constraint equation, prolong in each direction, apply commutators, repeat.
- Generates the set of conditions for the constraint to be consistent with given equations (with C.B. Collins).
- Recent work in computer algebra builds on classical Riquier-Janet theory, providing tools for differential ideals (e.g. Hubert, Reid, Schwartz).

Properties of the EM Field: E and B parallel to e_1

- Earlier work assumed $\tau_{ab} = \mathrm{diag}(\tau, -\tau, \tau, \tau)$ where $\tau = \frac{1}{2}(E^2 + B^2)$ with $[E, 0, 0]$ and $[B, 0, 0]$ the electric and magnetic fields in the rest space of u^a.

- In this case, for general LRS, Maxwell's equations are

$$\begin{aligned}
\partial_1 E &= -2\omega B + 2aE + \epsilon \\
\partial_1 B &= 2\omega E + 2aB \\
\partial_0 E &= -2\beta E - kB \\
\partial_0 B &= -2\beta B + kE
\end{aligned}$$

and $\partial_2 E = \partial_3 E = \partial_2 B = \partial_3 B = 0$.

- However F_{ab} is not covariantly defined...

Properties of the EM Field, more generally

- We have $\tau_{ab} = \frac{1}{4} g_{ab}(F_{cd}F^{cd}) - F_{ac}F_b{}^c$ as

$$\begin{aligned}
\tau_{00} &= \frac{1}{2}(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}) \\
\tau_{0\alpha} &= -(\mathbf{E}\times\mathbf{B})_\alpha \\
\tau_{\alpha\beta} &= -\left(E_\alpha E_\beta + B_\alpha B_\beta\right) + \frac{1}{2} g_{\alpha\beta}(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B})
\end{aligned}$$

- By LRS we have

$$\tau_{ab} = \begin{bmatrix} \tau_{00} & \tau_{01} & 0 & 0 \\ \tau_{10} & \tau_{11} & 0 & 0 \\ 0 & 0 & \tau_{22} & 0 \\ 0 & 0 & 0 & \tau_{33} \end{bmatrix}$$

where $\tau_{22} = \tau_{33}$ as well as $\tau_{01} = \tau_{10}$.

Properties of the EM Field, more generally

- This form of τ_{ab} gives

$$\begin{aligned}
\tau_{22} = \tau_{33} \quad &\Leftrightarrow & E_2^2 + B_2^2 = E_3^2 + B_3^2 \qquad (1)\\
\tau_{12} = 0 \quad &\Leftrightarrow & E_1 E_2 + B_1 B_2 = 0 \qquad (2)\\
\tau_{13} = 0 \quad &\Leftrightarrow & E_1 E_3 + B_1 B_3 = 0 \qquad (3)\\
\tau_{23} = 0 \quad &\Leftrightarrow & E_2 E_3 + B_2 B_3 = 0 \qquad (4)
\end{aligned}$$

  and ${\tau^a}_a = 0$ is identically satisfied.

- We consider two cases:
  Case 1: $E_2 = 0$ and $B_2 = 0$
  Case 2: $E_2 \neq 0$ or $B_2 \neq 0$.

Case 1: $E_2 = 0$ **and** $B_2 = 0$

- Equation (1) $\Rightarrow E_3 = B_3 = 0$. Previously known case.
- $\tau_{ab} = \text{diag}(\tau, -\tau, \tau, \tau)$
- $\tau = \frac{1}{2}(\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B}) = \frac{1}{2}(E_1^2 + B_1^2)$.

Case 2: $E_2 \neq 0$ or $B_2 \neq 0$

- Consider separately cases with one or both of $E_2 \neq 0$, $B_2 \neq 0$.
- In each case, equations (1-4) imply

$$E_\alpha = [0, E_2, E_3] \qquad B_\alpha = [0, \pm E_3, \mp E_2].$$

$$\tau_{ab} = \begin{bmatrix} \tau & \tau & 0 & 0 \\ \tau & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\tau = \frac{1}{2}(\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B}) = E_2^2 + E_3^2$.

Maxwell's Equations

In Case 2 with one choice of sign we have:

- $E_\alpha = [0, E_2, E_3] \qquad B_\alpha = [0, E_3, -E_2]$
- This gives Maxwell's equations as

$$
\begin{aligned}
(\partial_0 + \partial_1)E_2 &= -E_2\,(\alpha + \beta + a - \dot{u}) \qquad & \partial_2 E_2 &= -\partial_3 E_3 + E_2 s \\
(\partial_0 + \partial_1)E_3 &= -E_3\,(\alpha + \beta + a - \dot{u}) \qquad & \partial_2 E_3 &= \quad \partial_3 E_2 + E_3 s \\
\epsilon &= 0
\end{aligned}
$$

It is straightforward to show $\partial_0 \tau = -\partial_1 \tau - \tau\,(\alpha + \beta + a - \dot{u})$ and

$$
\begin{aligned}
\partial_2 E_2 &= 0 \qquad & \partial_2 E_3 &= 0 \\
\partial_3 E_2 &= -s E_3 \qquad & \partial_3 E_3 &= s E_2
\end{aligned}
$$

- In LRS II, the [2, 3] commutator implies $r = 0$ so $\partial_2 s = s^2$.

Physical Significance

- In Case 2 we have $\mathbf{E} \times \mathbf{B}$ parallel to $\mathbf{e}_1$.
- This is the Poynting vector, describing the energy flux of the field in the cov. defined rest space of u^a.

- In Case 1 we have trivially that $\mathbf{E} \times \mathbf{B} = 0$ is parallel to $\mathbf{e}_1$.

So it is *always* the case that $\mathbf{E} \times \mathbf{B}$ is parallel to $\mathbf{e}_1$.

Related Work

- Misner and Wheeler (Ann Phys 1957): If $\mathbf{F}$ is non-singular it is determined from the metric only up to duality transformations.

- Henneaux (J Math Phys 1984): EM fields invariant up to a duality rotation under a group H of isometries.
- Considers $\mathbf{F}$ such that $h^* \, \mathbf{F} = \cos\alpha(h)\mathbf{F} + \sin\alpha(h)\mathbf{F}^*$ for all elements $h \in H$, $h^* \, \mathbf{F}$ is pullback of $\mathbf{F}$, $\mathbf{F}^*$ is dual 2-form.
- Examined Bianchi models whose source is an EM field sharing the symmetry of the metric up to a duality transformation. G_3 on space-like hypersurfaces.

Conclusions

- It is well known that symmetries of the metric are not always symmetries of the EM field. It appears that LRS spacetimes admit such solutions.
- If there is a covariantly defined time-like vector field u^a (e.g. fluid flow), then the Poynting vector field is covariantly defined in the rest space of u^a and is subject to the same symmetry conditions as all other covariantly defined quantities.
- This limits the admisible transformations on $\mathbf{F}$.