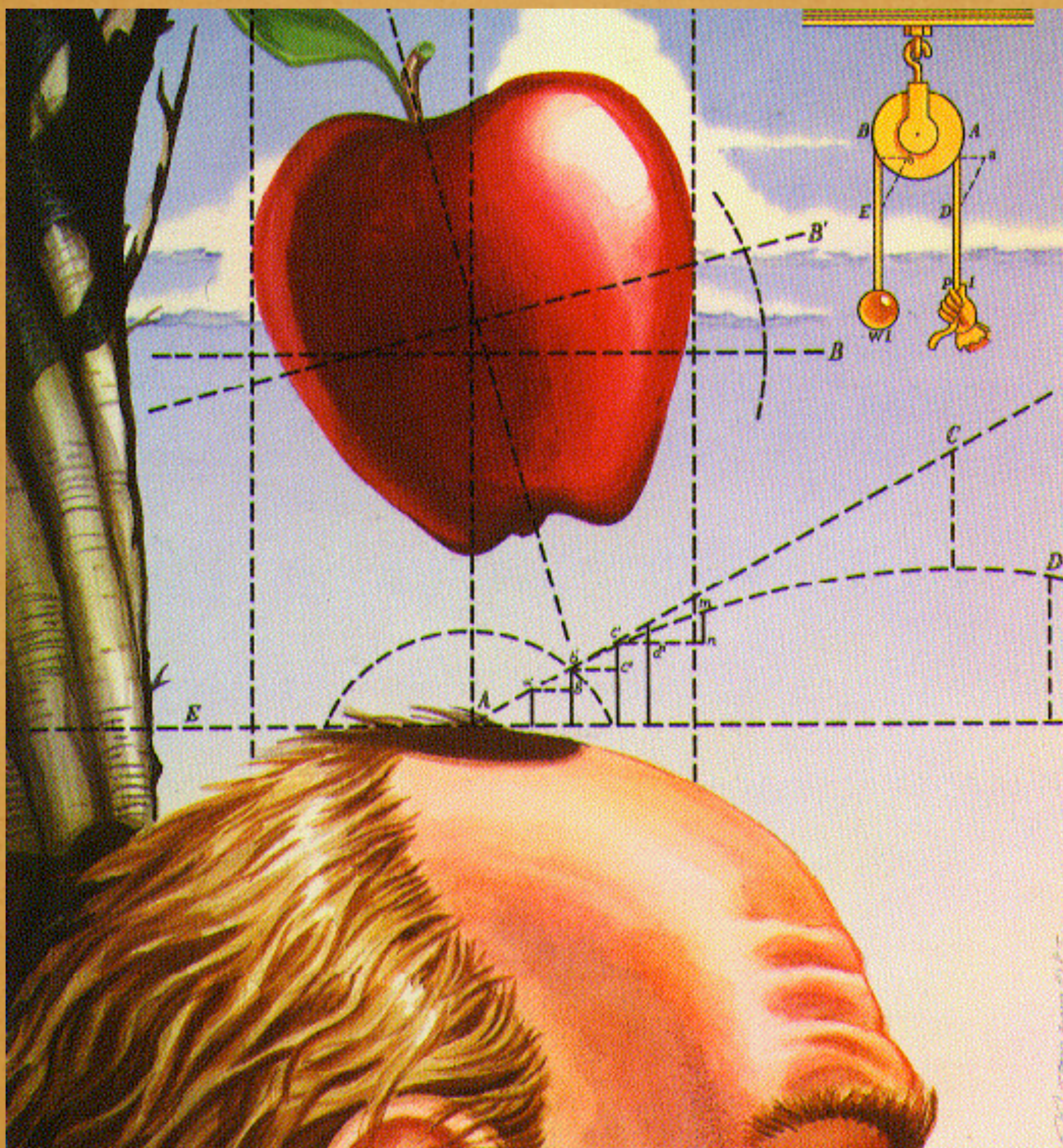
The Evolution of Cosmological Perturbations in F(R)-gravity



Sante Carloni

Cosmo 2008
Madison (WI)

Dark Energy (DE)

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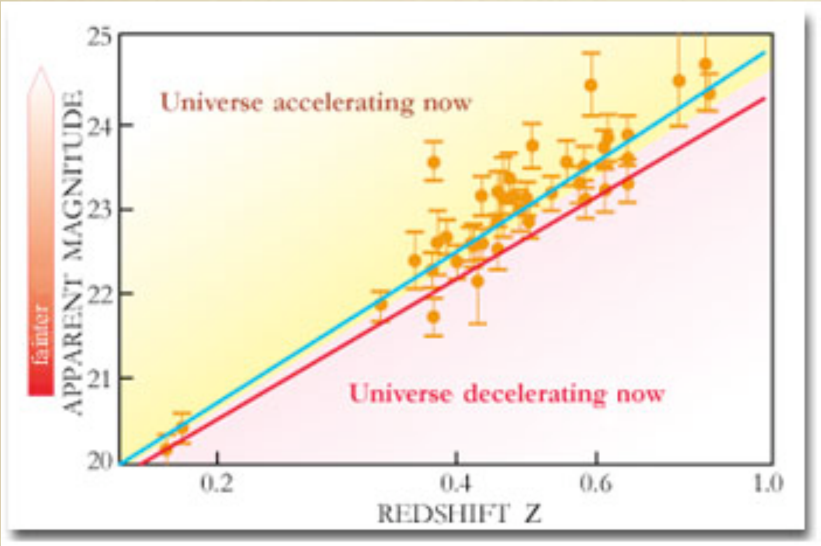

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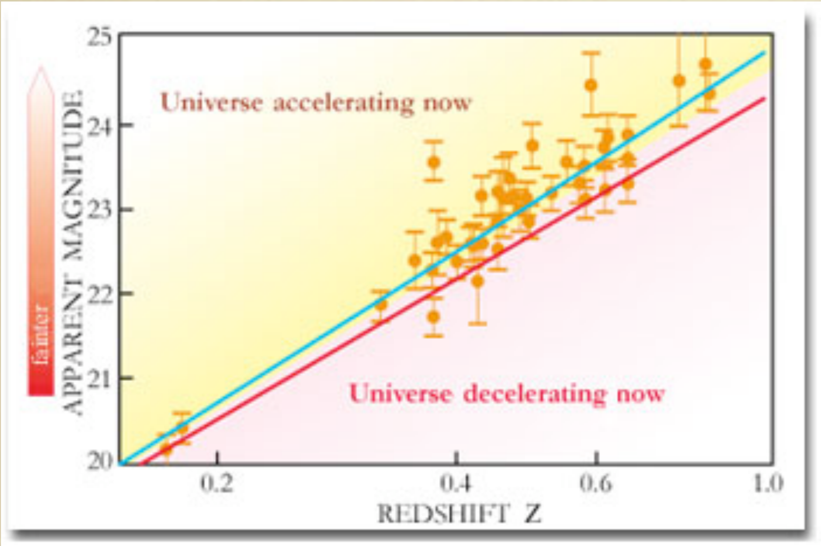

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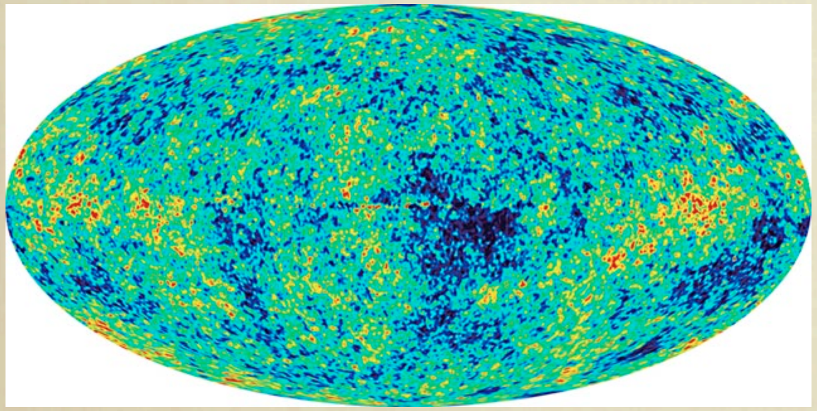

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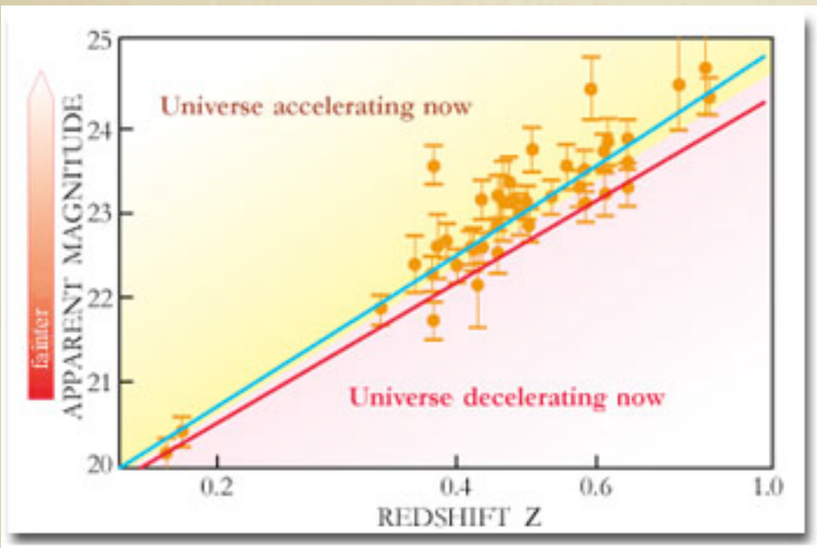

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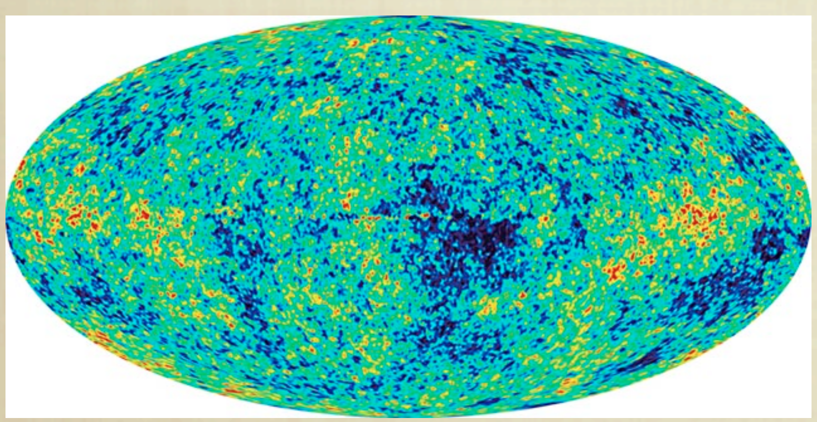

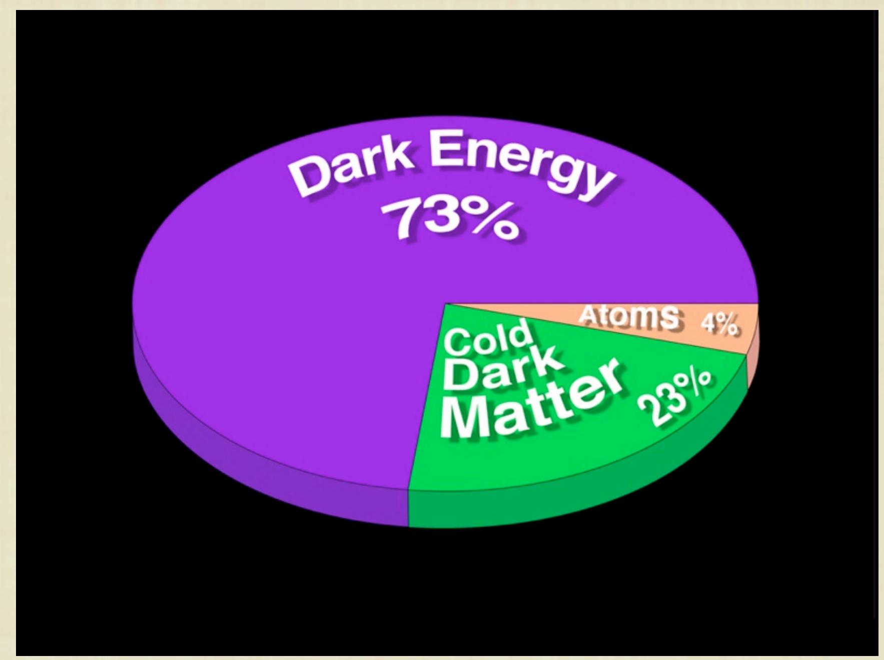

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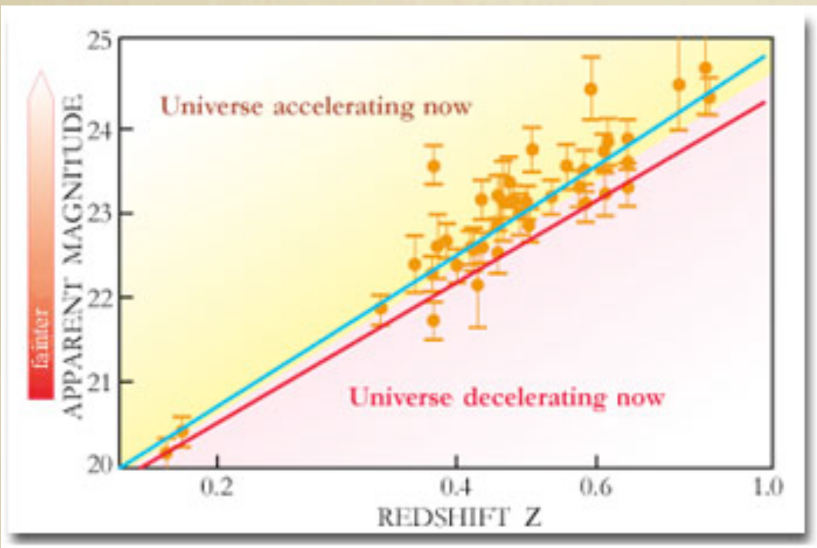

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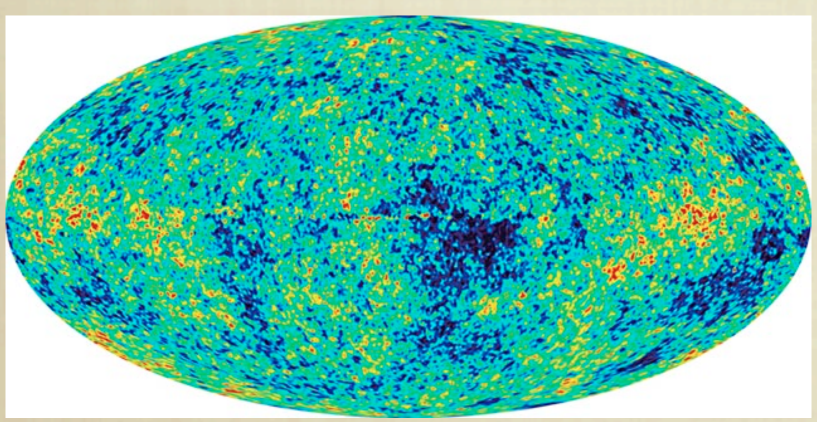
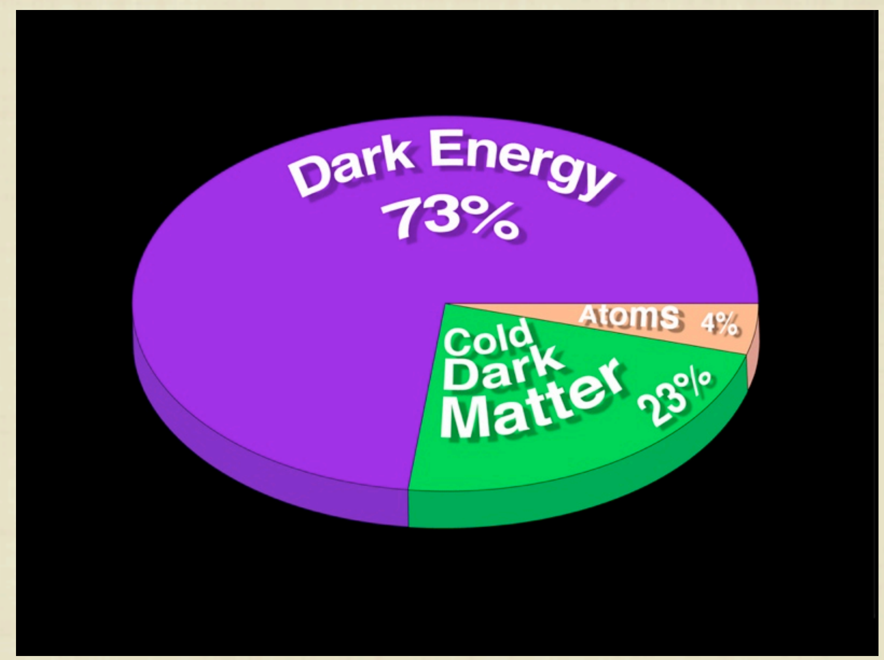


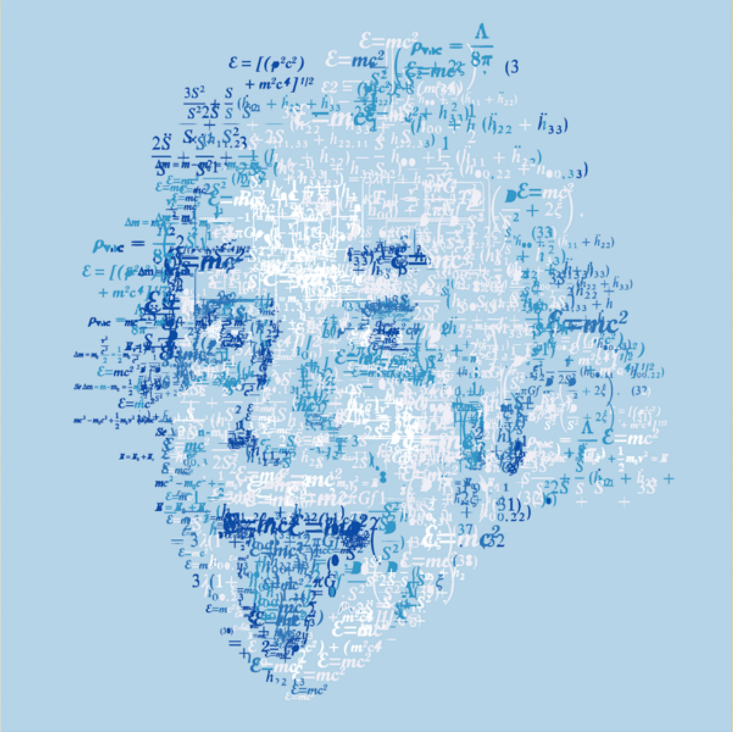


Many different models for dark energy have been proposed so far. We will focus on the ones based on **Fourth Order Gravity (FOG)**

Fourth Order Gravity

In homogeneous and isotropic spacetimes a general action for fourth order gravity in presence of matter is

$$\mathcal{A} = \int d^4x \sqrt{-g}\, f(R) + \int d^4x \sqrt{-g}\, L_M \ .$$



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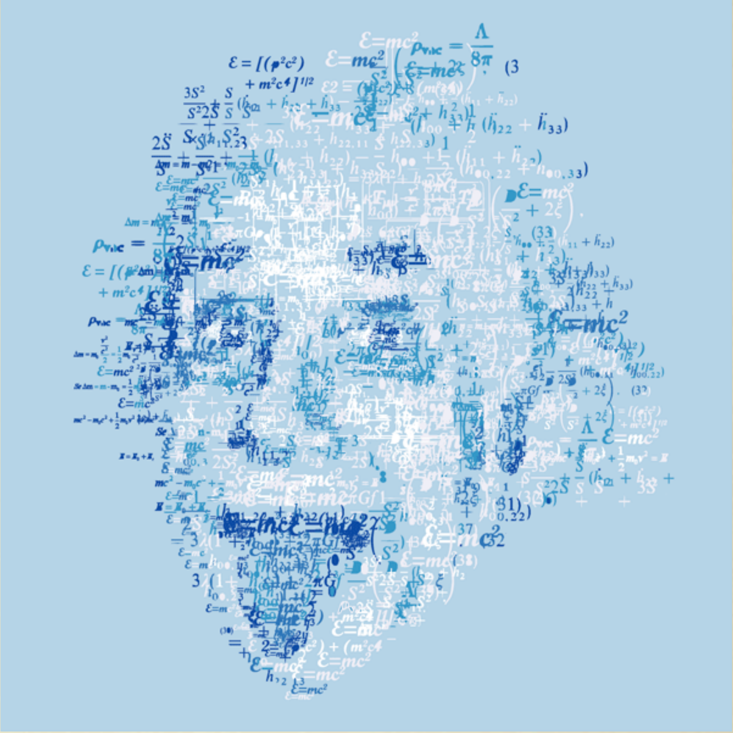

$$\mathcal{A} = \int d^4x\,\sqrt{-g}\,f(R) + \int d^4x\,\sqrt{-g}\,L_M\ .$$

varying with respect to the metric gives

$$f'(R)R_{ab} - \frac{1}{2}f(R)g_{ab} = f'(R)^{;cd}\left(g_{ca}g_{db} - g_{cd}g_{ab}\right) + \tilde{T}^M_{ab}\ ,$$

where

$$\tilde{T}^M_{ab} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_M)}{\delta g_{ab}}$$

and the “prime” denotes the derivative with respect to the Ricci scalar.

DE and FOG

Why Fourth order gravity is an interesting model for DE?

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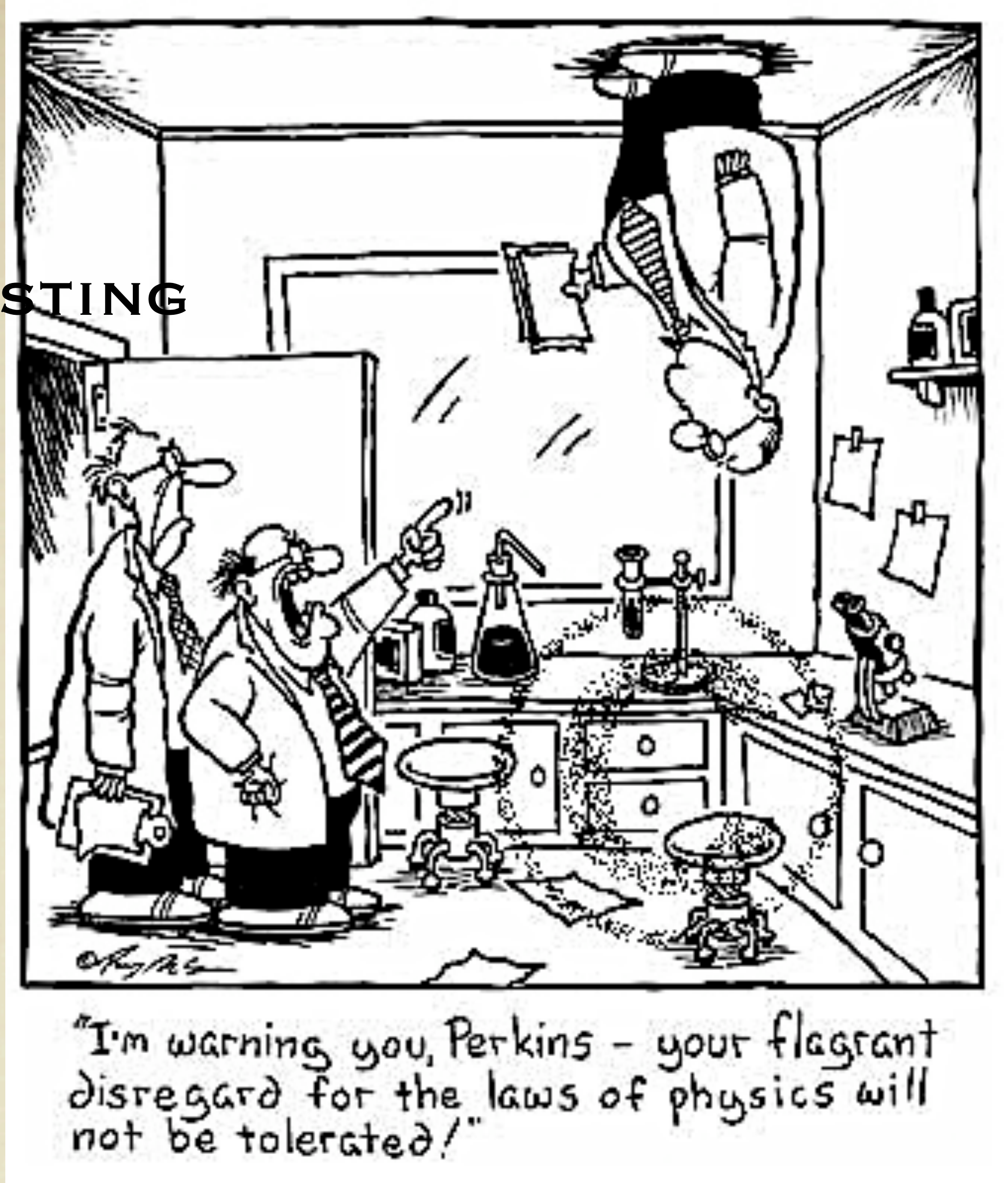
- Differently from GR, they admit naturally cosmological solutions characterized by accelerated expansion i.e. the footprint of DE

- They are recovered as low energy limit of more fundamental schemes like M-theory, supergravity etc.



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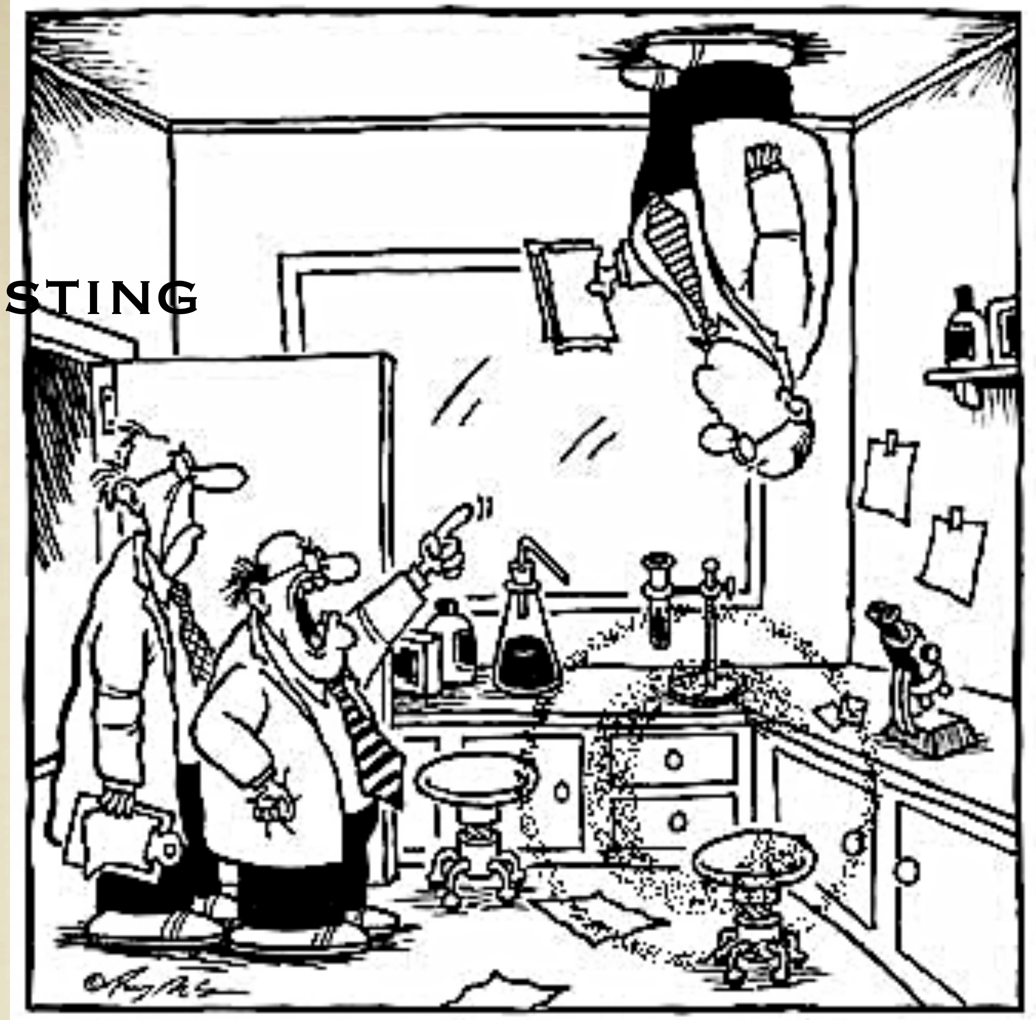

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Unfortunately, due to their high degree of non linearity, this kind of theories are particularly difficult to deal with.

So one needs to develop new techniques to be able to unfold their properties

1+3 covariant approach

Given the vector field associated to a time-like flow in the model:

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1+3 Equations
$(\Theta, \sigma_{ab}, \dot{u}_a, \omega_{ab}, \mu^i, p^i)$

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This approach has many advantages:

- its variables have a clear physical meaning at any stage of the calculations and are gauge invariant
- the treatment of both the exact and the linearized theory is considerably simplified
- the same variables can be used in perturbing different models e.g anisotropic spacetimes etc.
- it is easily adaptable to alternative gravity

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Scalar Perturbation variables

Scalar Perturbation variables

The natural set of inhomogeneity variables associated with the spherical collapse in GR are:

$$\Delta_m = S^2 \frac{\tilde{\nabla}^2 \mu}{\mu} \qquad Z = S^2 \tilde{\nabla}^2 \Theta \qquad C = S^4 \tilde{\nabla}^2 \tilde{R}$$

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$$\mathcal{R} = S^2 \tilde{\nabla}^2 R \qquad \Re = S^2 \tilde{\nabla}^2 \dot{R}$$

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we obtain

$$\ddot{\Delta}_m^{(k)} + \left[\left(\frac{2}{3} - w\right)\Theta - \frac{\dot{R} f''}{f'}\right]\dot{\Delta}_m^{(k)} - \left[w\frac{k^2}{S^2} - w(3p^R + \mu^R) - \frac{2w\dot{R}\Theta f''}{f'} - \frac{\left(3w^2 - 1\right)\mu}{f'}\right]\Delta_m^{(k)}$$
$$= \frac{1}{2}(w+1)\left[2\frac{k^2}{S^2}f'' - 1 + \left(f - 2\mu + 2\dot{R}\Theta f''\right)\frac{f''}{f'^2} - 2\dot{R}\Theta\frac{f^{(3)}}{f'}\right]\mathcal{R}^{(k)} - \frac{(w+1)\Theta f''}{f'}\dot{\mathcal{R}}^{(k)} \, ,$$

$$f''\ddot{\mathcal{R}}^{(k)} + \left(\Theta f'' + 2\dot{R} f^{(3)}\right)\dot{\mathcal{R}}^{(k)} - \left[\frac{k^2}{S^2}f'' + 2\frac{K}{S^2}f'' + \frac{2}{9}\Theta^2 f'' - (w+1)\frac{\mu}{2f'}f'' - \frac{1}{6}(\mu^R + 3p^R)f''\right.$$
$$\left. - \frac{f'}{3} + \frac{f}{6f'}f'' + \dot{R}\Theta\frac{f''^2}{6f'} - \ddot{R}f^{(3)} - \Theta f^{(3)}\dot{R} - f^{(4)}\dot{R}^2\right]\mathcal{R}^{(k)} = -\left[\frac{1}{3}(3w-1)\mu\right.$$
$$\left. + \frac{w}{1+w}\left(f^{(3)}\dot{R}^2 + (p^R + \mu^R)f' + \tfrac{7}{3}\dot{R}\Theta f'' + \ddot{R}f''\right)\right]\Delta_m^{(k)} - \frac{(w-1)\dot{R}f''}{w+1}\dot{\Delta}_m^{(k)}$$

Perturbation equations

We can then derive the evolution equations for these variables. Using the covariant harmonics defined by

$$\tilde{\nabla}^2 Q = -\frac{k^2}{S^2} Q \,,$$

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$$\begin{aligned}
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\end{aligned}$$

Note the k structure of these equations. It will be important for our final results.

A simple example...

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[\chi R^n + L_M \right] \ ,$$

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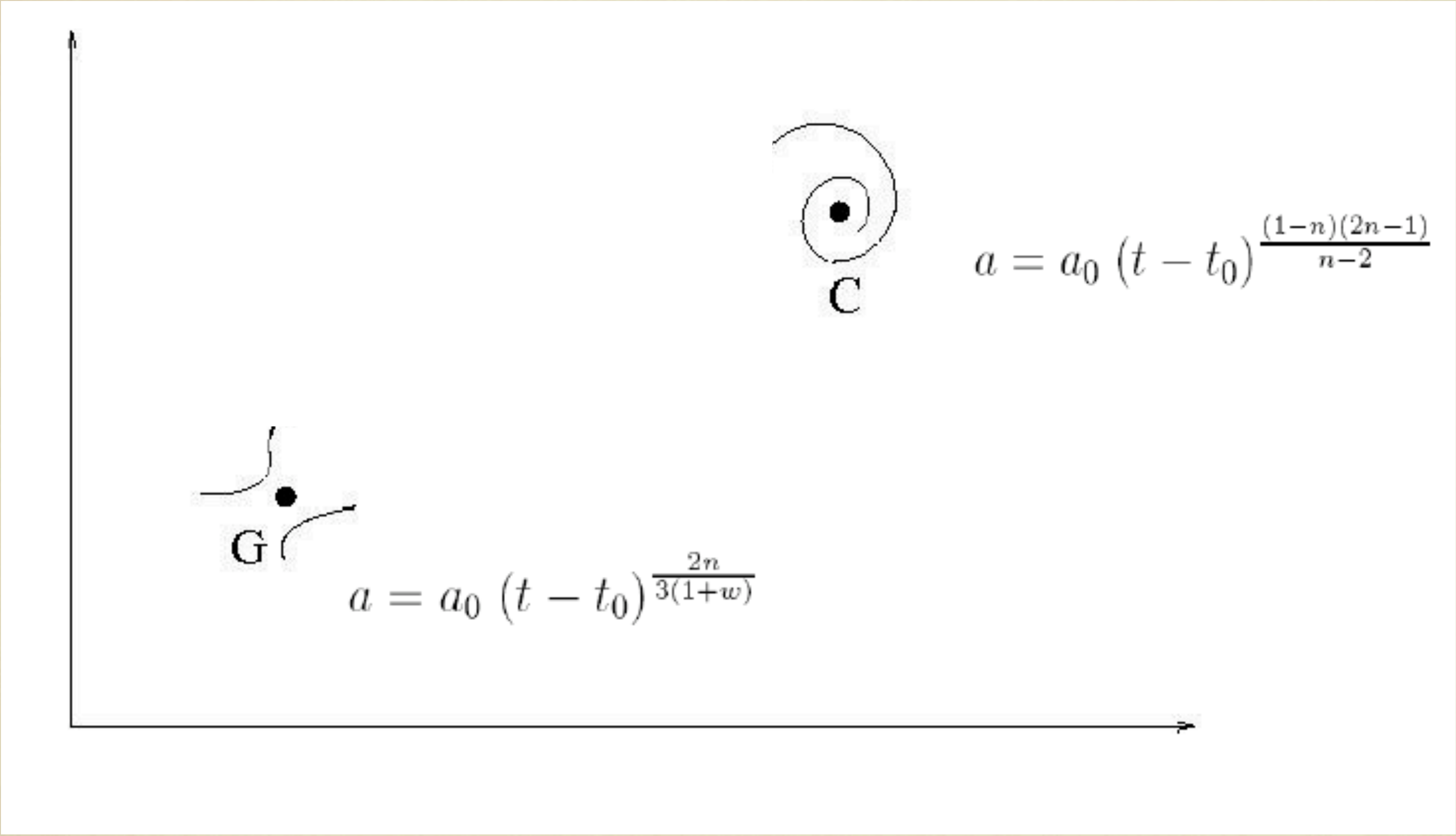

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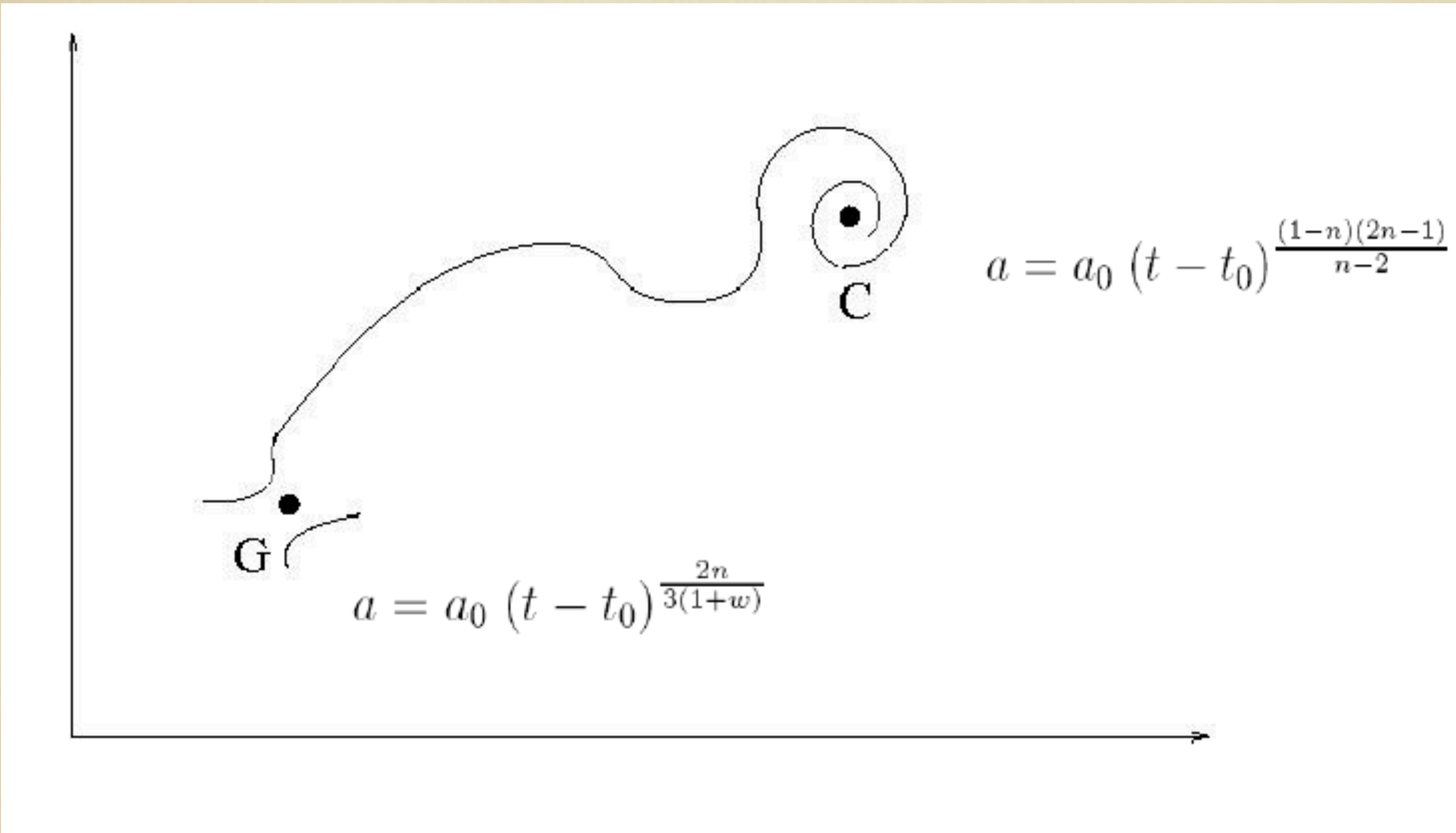

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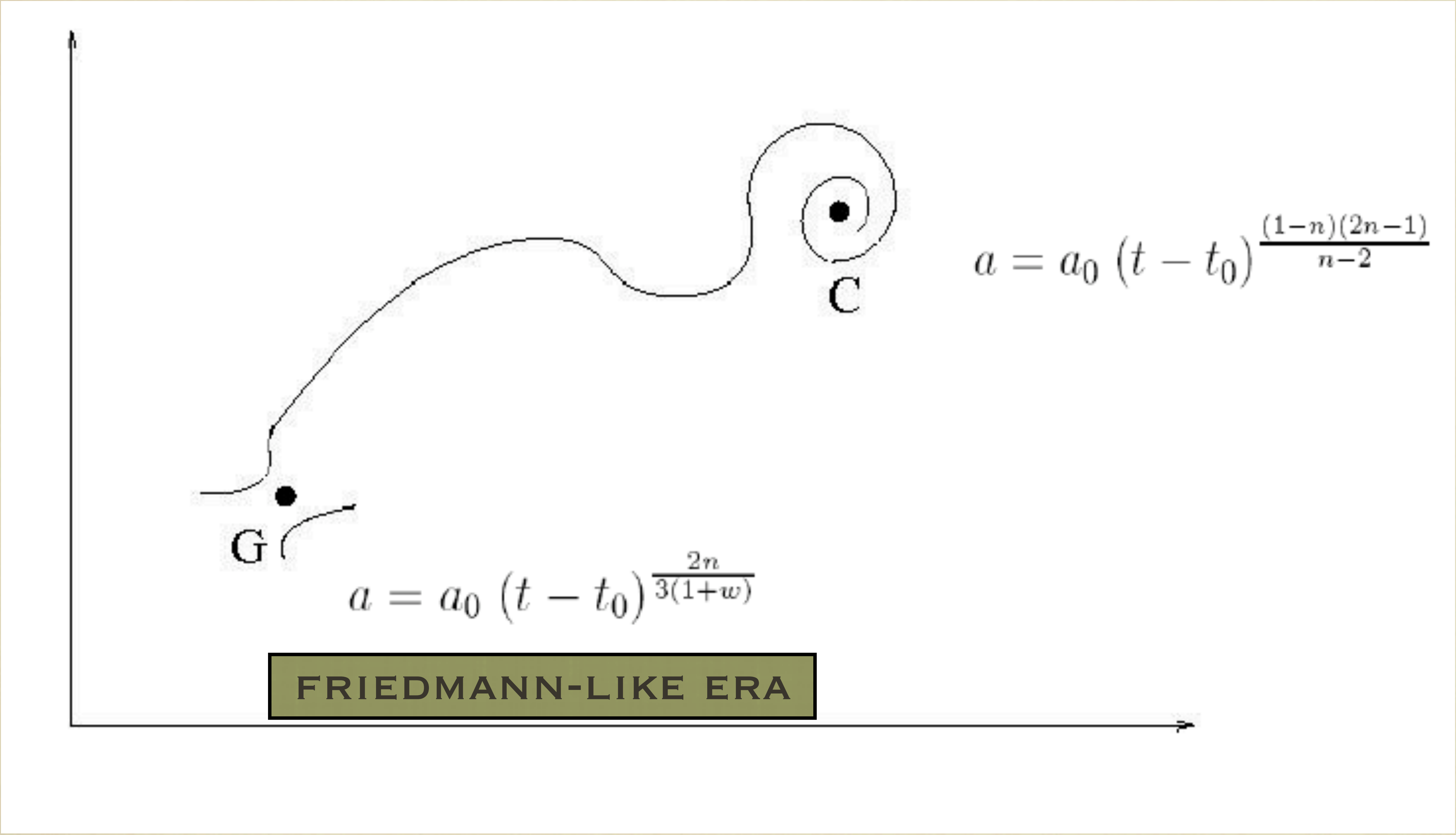

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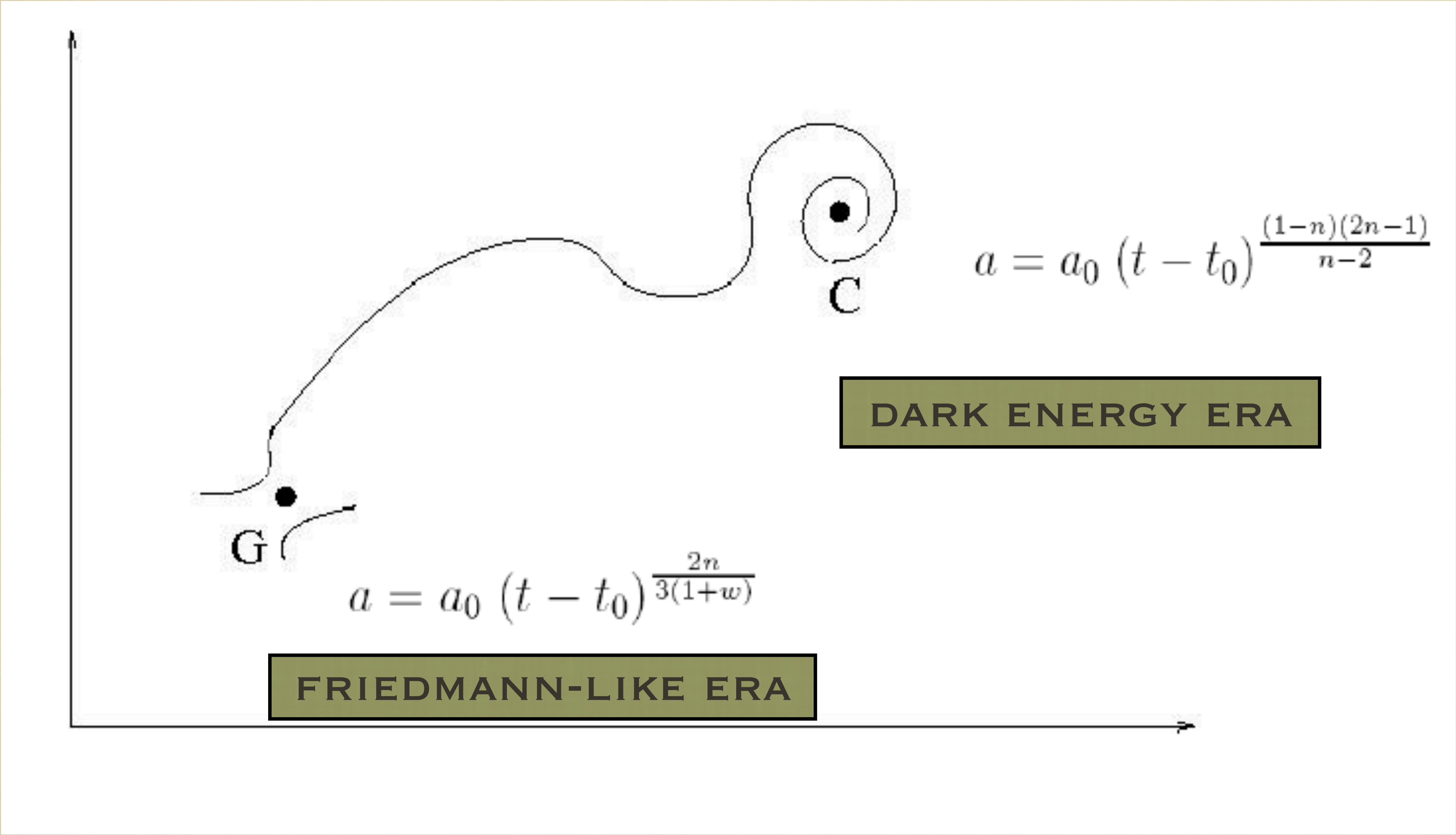

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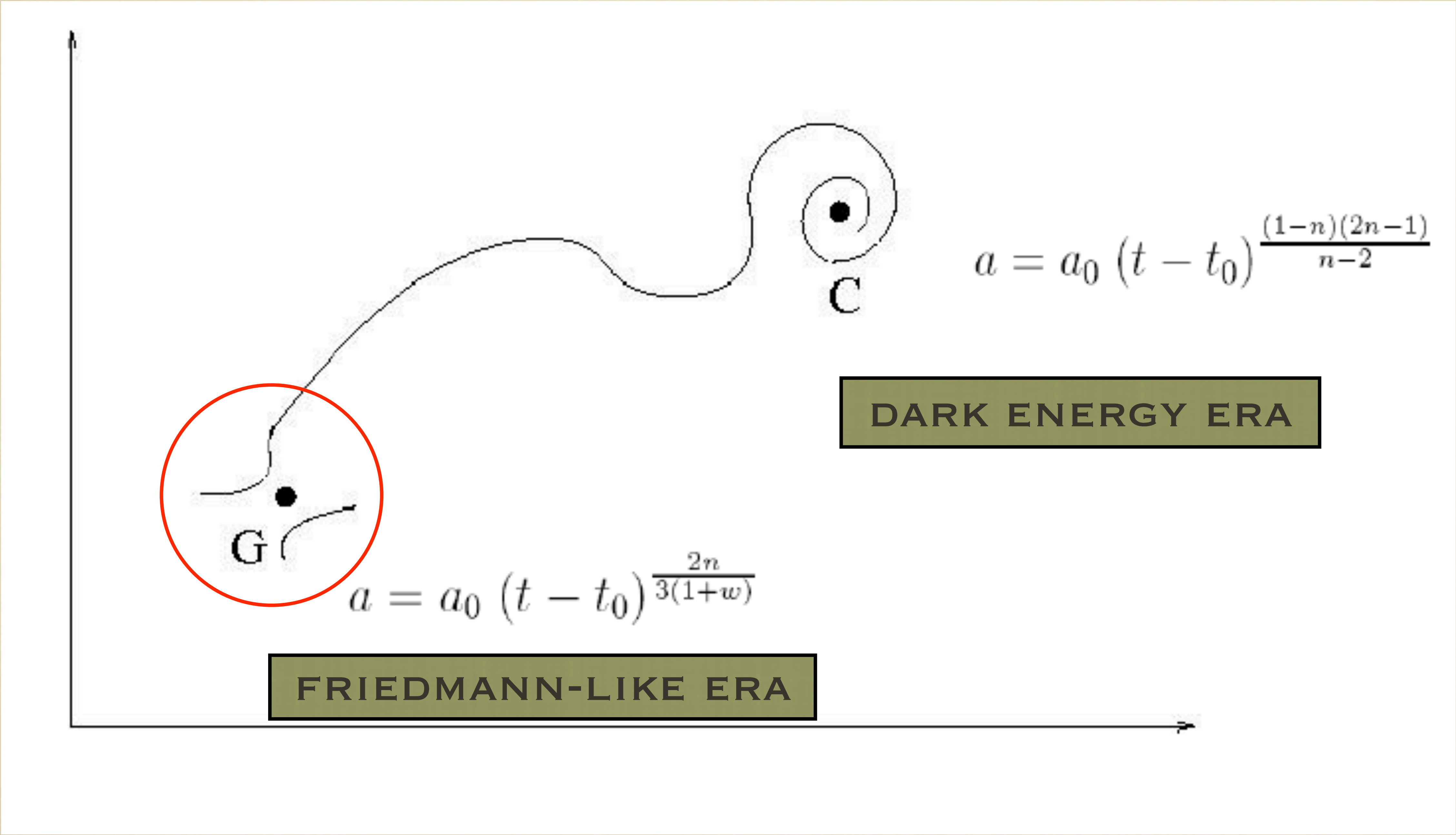

Let us investigate the behavior of the perturbations around this point.

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Large-scale density perturbations

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$$\Delta_m = K_1 t^{-1} + K_2 t^{\alpha_+|_{w=0}} + K_3 t^{\alpha_-|_{w=0}} - K_4 \frac{\mathcal{C}_0}{S_0^2} t^{2-\frac{4n}{3}} \ ,$$

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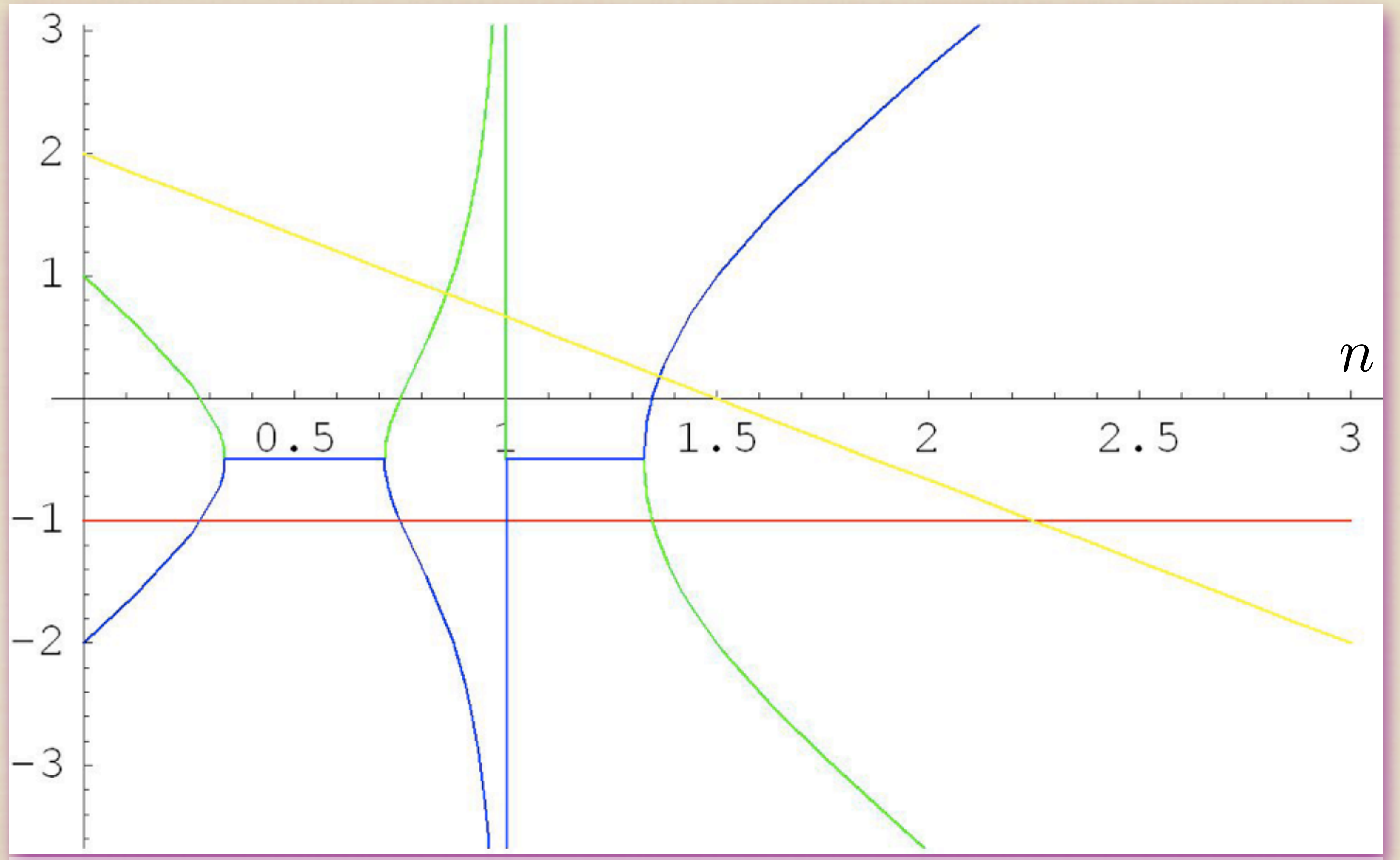

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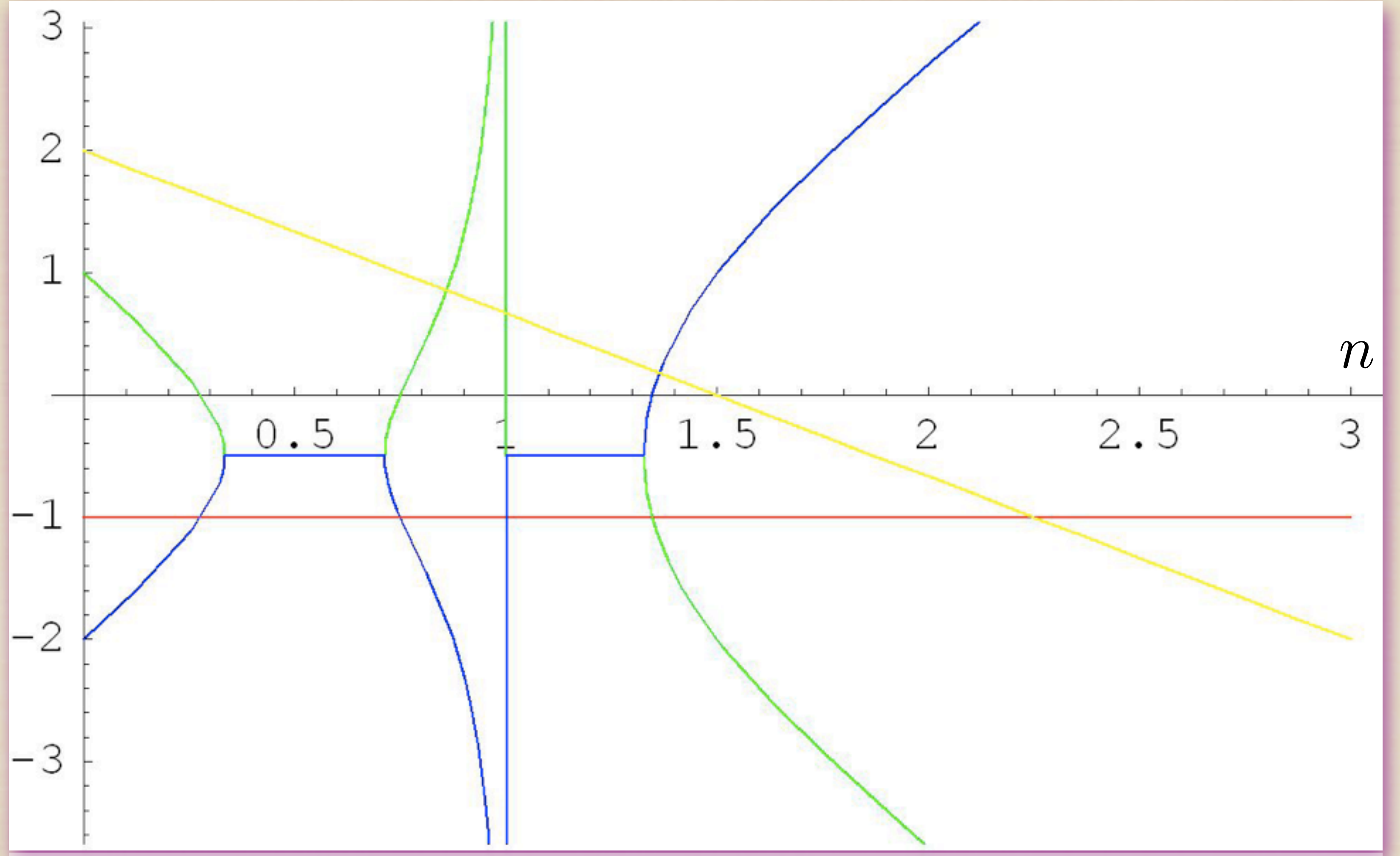

- For 0.33 < n < 0.71 and 1 < n < 1.32 the modes become oscillatory with negative real part
- Only for 1.32 < n < 1.43 do the modes grow at a rate less than the standard GR growing mode

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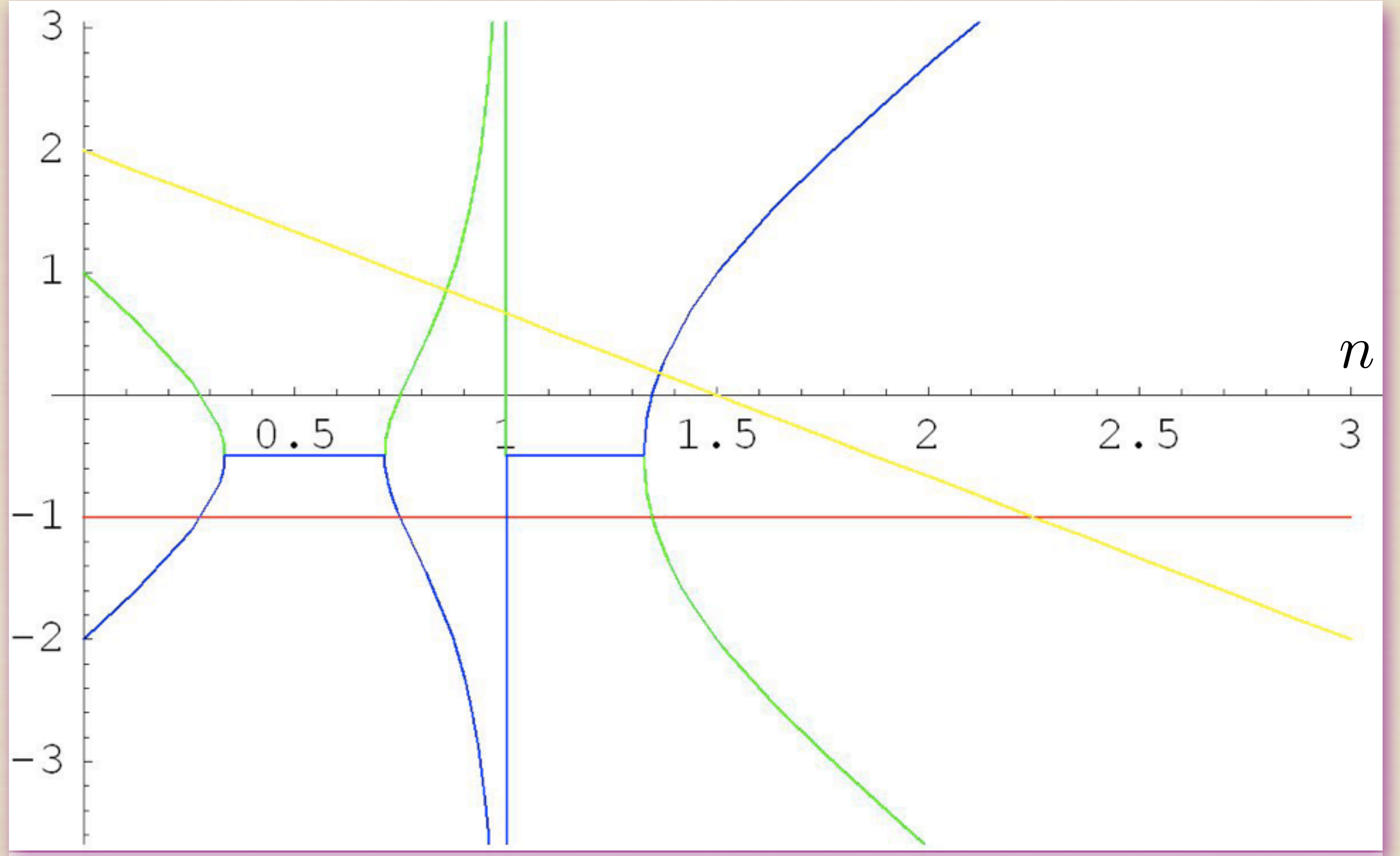

The long wavelength perturbations grow for all values of n, even for an accelerated background!

The Matter Power Spectrum

An important quantity to characterized the small scale perturbations in the power spectrum

$$\langle \Delta_m(\mathbf{k}_1)\Delta_m(\mathbf{k}_2)\rangle = P(k_1)\delta(\mathbf{k}_1+\mathbf{k}_2)$$

This quantity tells us how the fluctuations of matter depend on the wavenumber at a specific time and carries informations on the amplitude of the perturbations.

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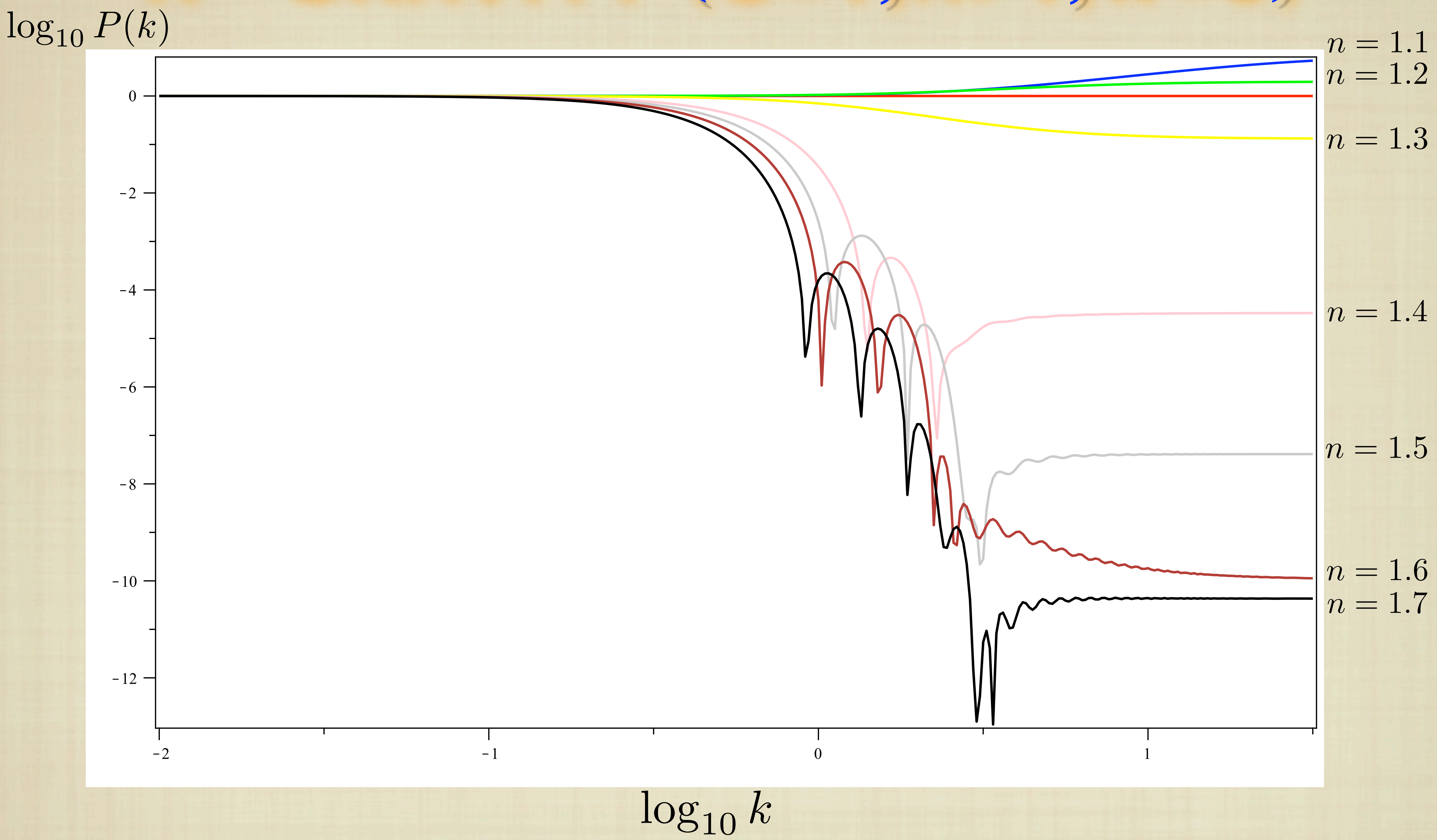
In GR the power spectrum on large scale is constant, while on small scales it is suppressed depending on the nature of the cosmological fluid(s).

The case of dust is special: perturbations are scale invariant.

The Matter Power Spectrum for R^n-gravity ($s=1, n>1, w=0$)

The Matter Power Spectrum for R^n-gravity (s=1,n>1,w=0)



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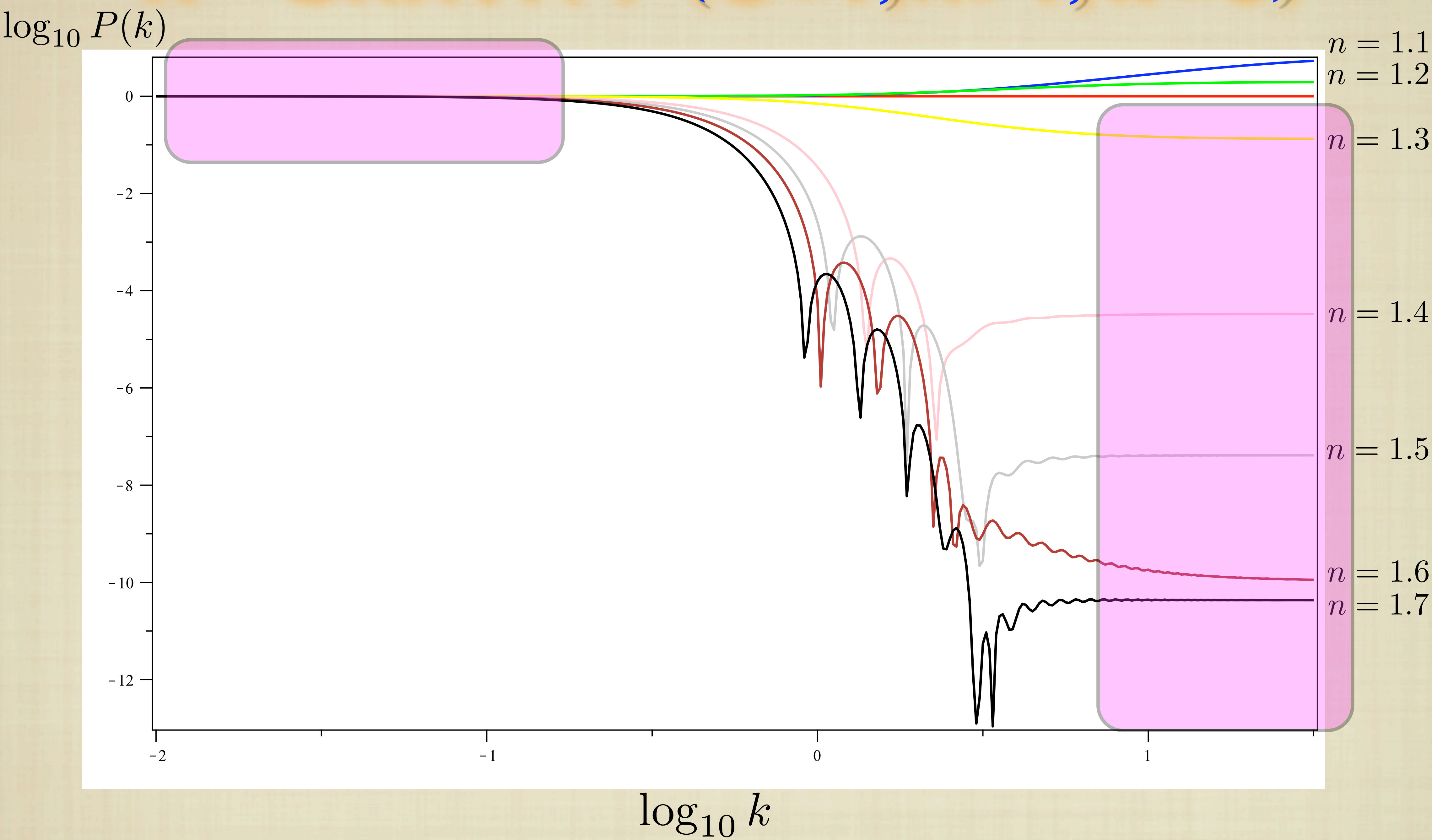

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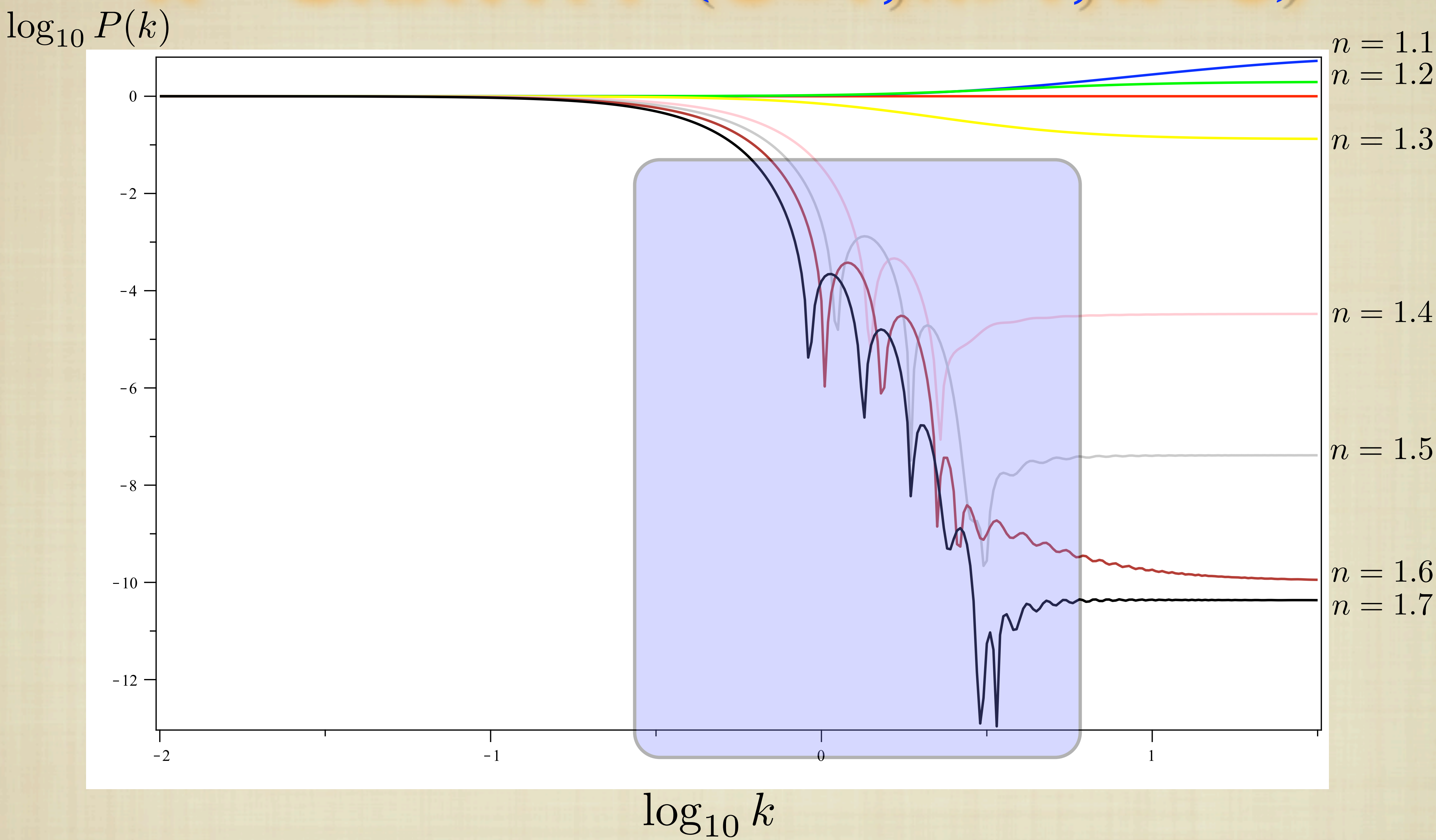

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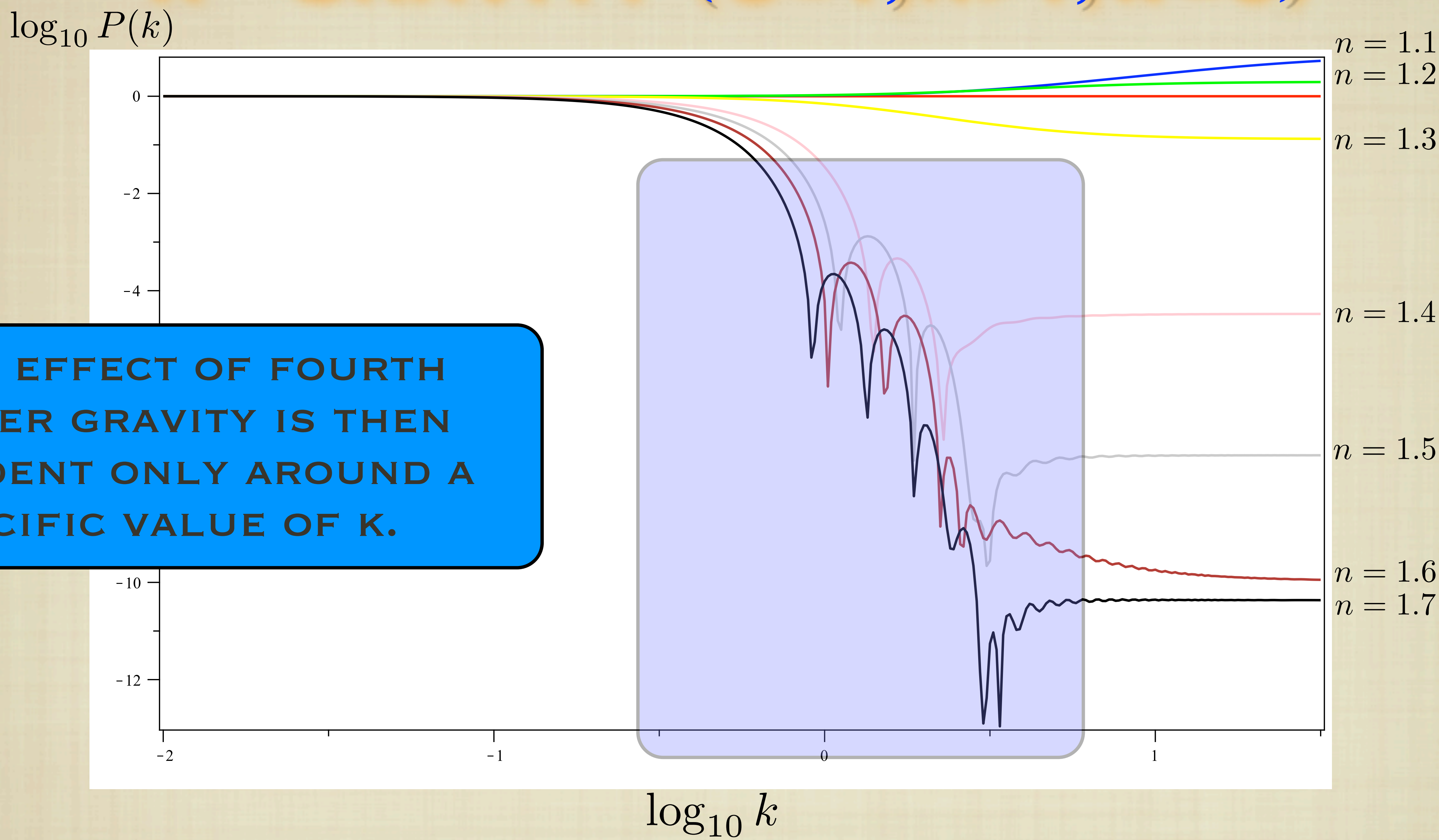

The effect of fourth order gravity is then evident only around a specific value of k.

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Evolution of P(k) [n=1.4, w =0]

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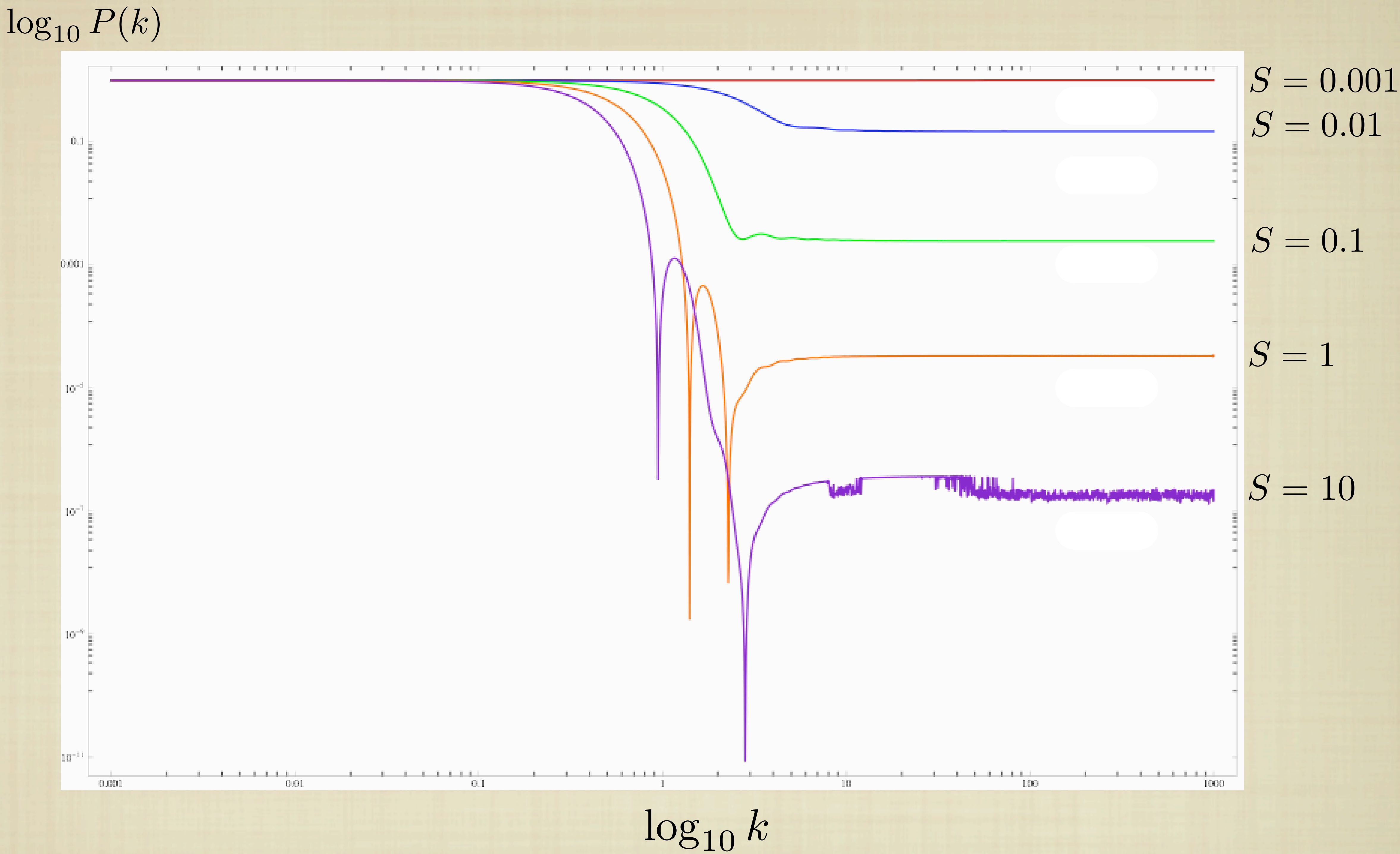

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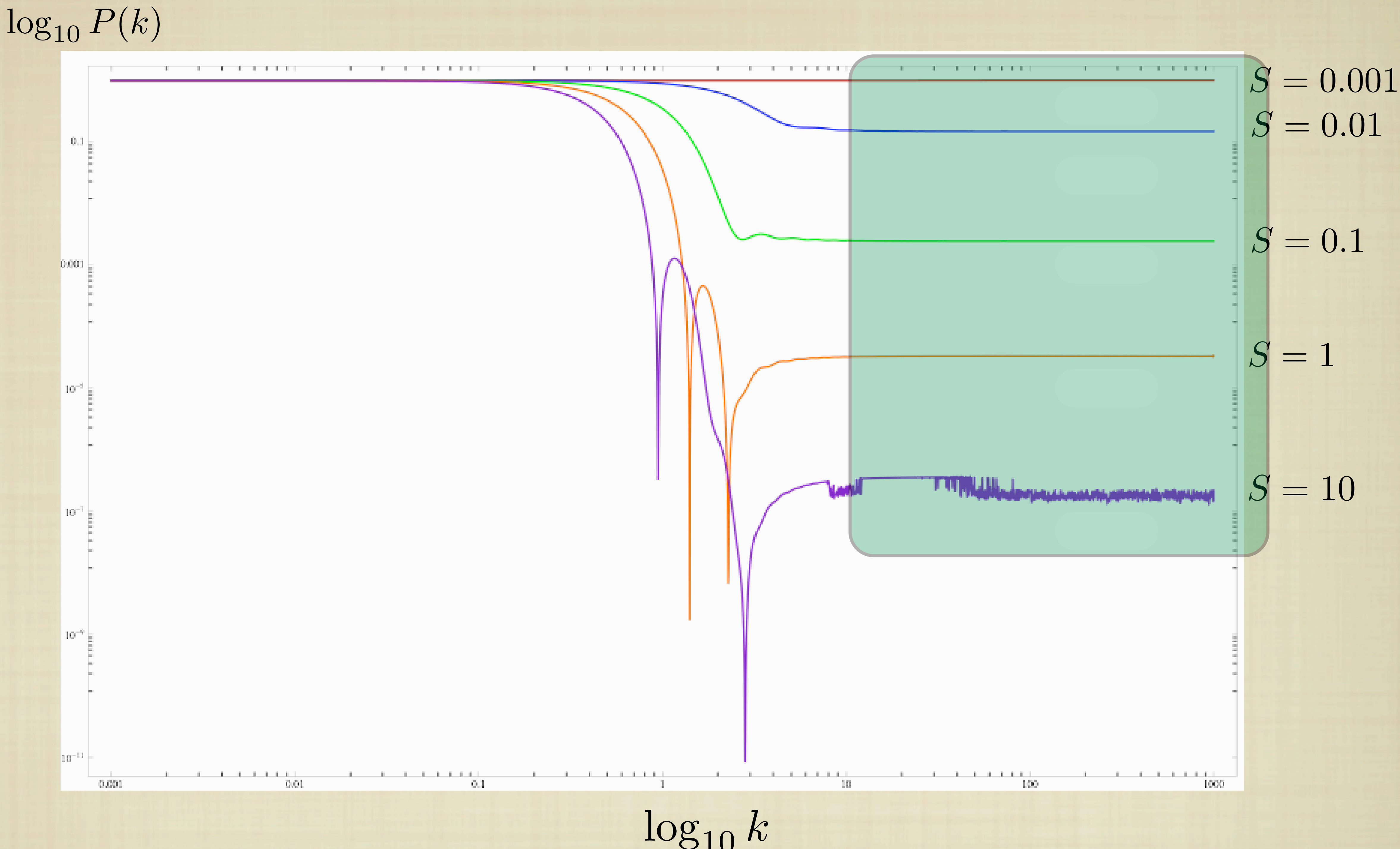

The small scale perturbations loose power in time (and the Large scale ones grow)

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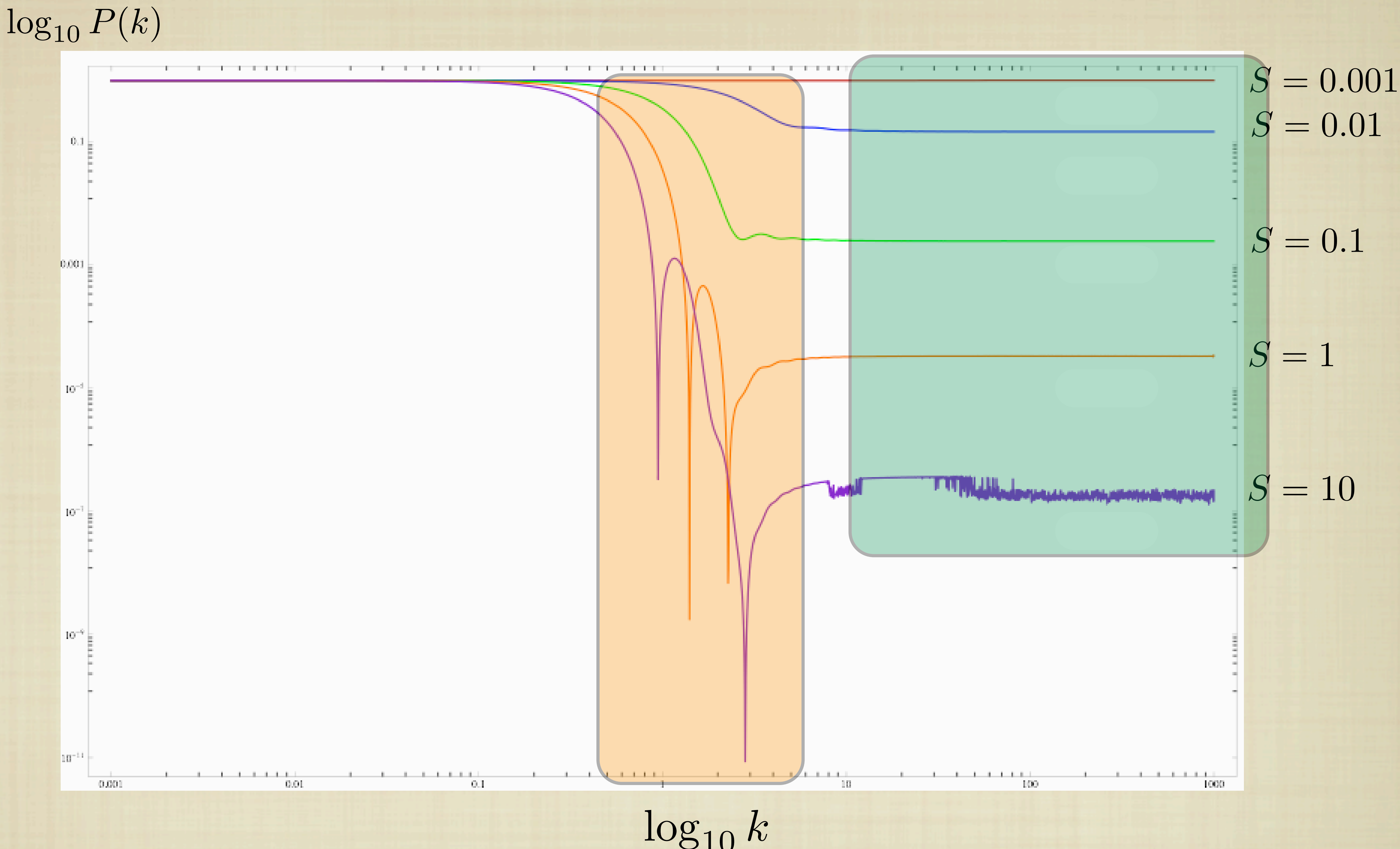

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Oscillations in the spectrum start to appear as the universe evolves.

The Case F(R)=R+αR^N

The action in this case reads

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Point	Coordinates (x, y, z, Ω)	Scale Factor
$\mathcal{A}$	$(0, 0, 0, 0)$	$a(t) = (t - t_0)$
$\mathcal{B}$	$(-1, 0, 0, 0)$	$a(t) = a_0\,(t - t_0)^{1/2}$
$\mathcal{C}$	$(-1 - 3w, 0, 0, -1 - 3w)$	$a(t) = (t - t_0)$
$\mathcal{D}$	$(1 - 3w, 0, 0, 2 - 3w)$	$a(t) = a_0\,(t - t_0)^{1/2}$
$\mathcal{E}^*$	$(0, -2, -1, 0)$	$\begin{cases} a(t) = a_0, \\ a(t) = a_0 \exp\left[\pm 2\sqrt{3}\alpha^\gamma (2 - 3n)^\gamma (t - t_0)\right], \end{cases}$ $\gamma = \frac{1}{2(1-n)}$
$\mathcal{F}$	$(2, 0, 2, 0)$	$a(t) = (t - t_0)$
$\mathcal{G}$	$(4, 0, 5, 0)$	$a(t) = a_0\,(t - t_0)^{1/2}$
$\mathcal{H}$	$(2(1 - n), 2n(n - 1), 2(1 - n), 0)$	$a(t) = \sqrt{1 - 2n(n - 1)}\,(t - t_0)$
$\mathcal{I}^*$	$\left(\frac{2(n-2)}{2n-1}, \frac{(5-4n)n}{2n^2-3n+1}, \frac{5-4n}{2n^2-3n+1}, 0\right)$	$a(t) = a_0\,(t - t_0)^{\frac{2n^2-3n+1}{2-n}}$
$\mathcal{L}$	$\left(-\frac{3(n-1)(w+1)}{n}, \frac{-4n+3w+3}{2n}, \frac{-4n+3w+3}{2n^2}, \frac{-2(3w+4)n^2+(9w+13)n-3(w+1)}{2n^2}\right)$	$a(t) = a_0\,(t - t_0)^{\frac{2n}{3(w+1)}}$

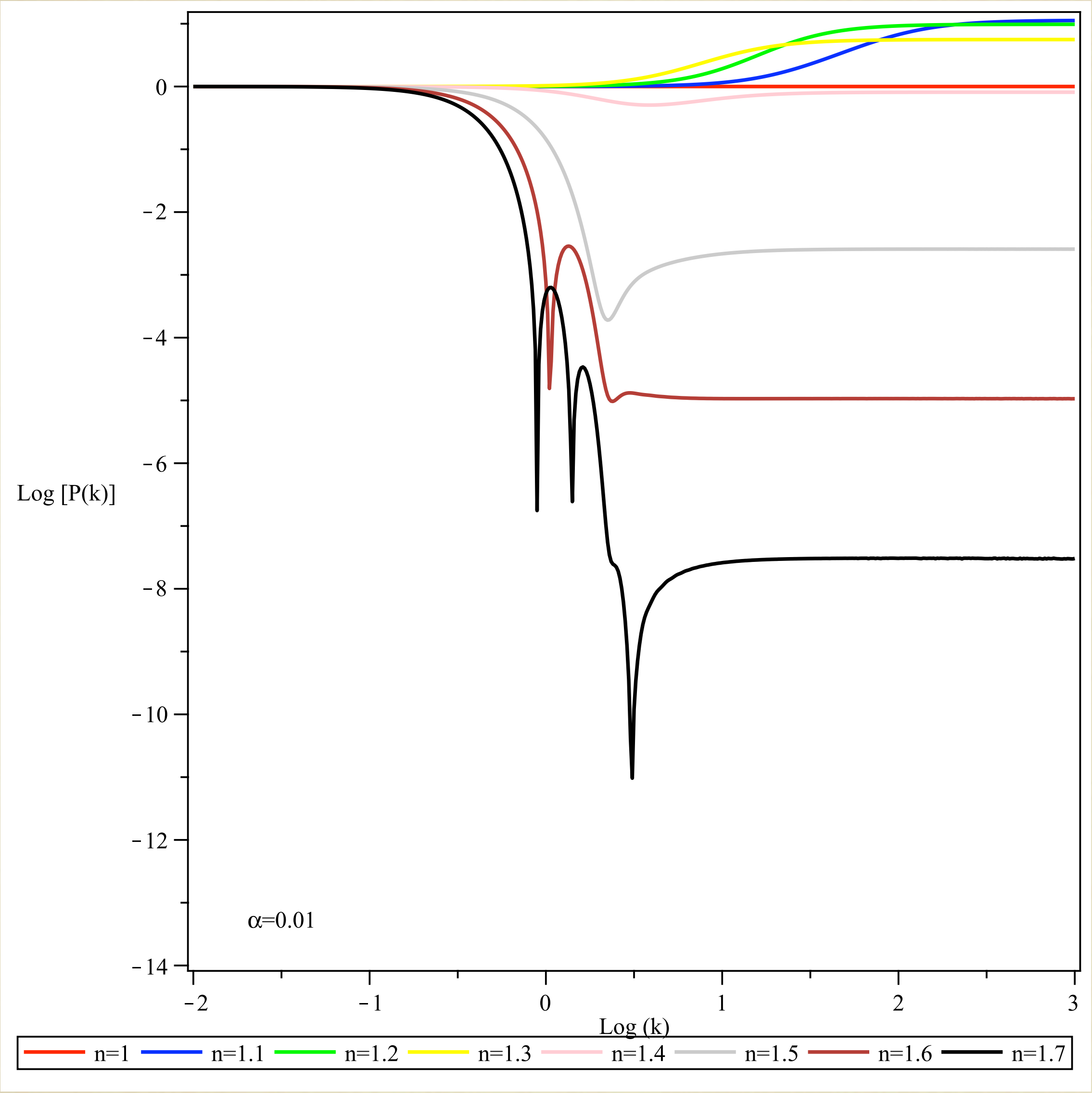
The Case F(R)=R+αR^N

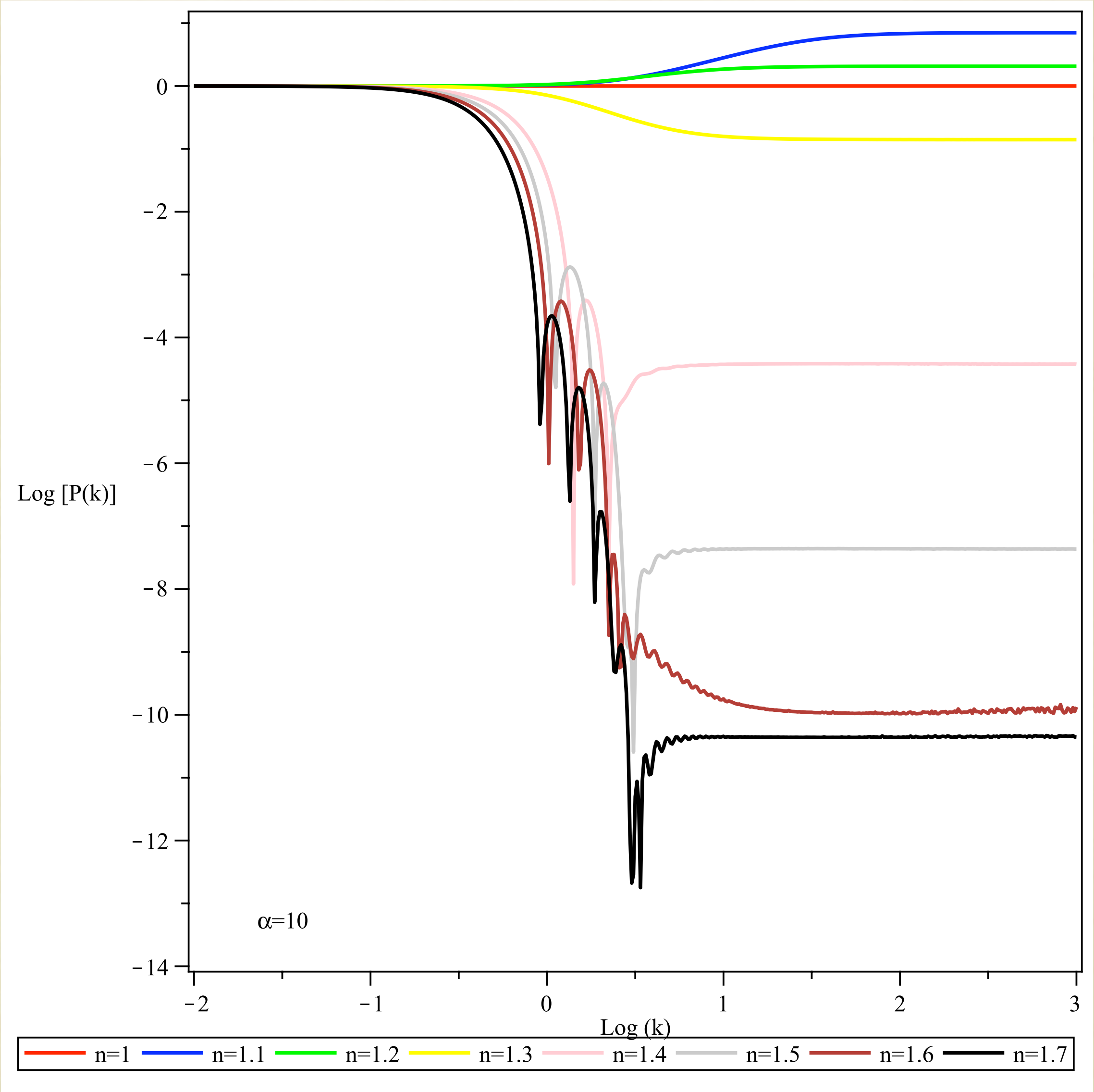
The action in this case reads

$$\mathcal{A} = \int d^4x\sqrt{-g}\,[R + \alpha R^n + L_M]$$

Point	Coordinates (x, y, z, Ω)	Scale Factor
$\mathcal{A}$	$(0,0,0,0)$	$a(t) = (t - t_0)$
$\mathcal{B}$	$(-1,0,0,0)$	$a(t) = a_0\,(t - t_0)^{1/2}$
$\mathcal{C}$	$(-1-3w,0,0,-1-3w)$	$a(t) = (t - t_0)$
$\mathcal{D}$	$(1-3w,0,0,2-3w)$	$a(t) = a_0\,(t - t_0)^{1/2}$
$\mathcal{E}^*$	$(0,-2,-1,0)$	$\begin{cases} a(t) = a_0, \\ a(t) = a_0 \exp\left[\pm 2\sqrt{3}\alpha^\gamma (2-3n)^\gamma (t - t_0)\right], \end{cases}$ $\gamma = \frac{1}{2(1-n)}$
$\mathcal{F}$	$(2,0,2,0)$	$a(t) = (t - t_0)$
$\mathcal{G}$	$(4,0,5,0)$	$a(t) = a_0\,(t - t_0)^{1/2}$
$\mathcal{H}$	$(2(1-n),2n(n-1),2(1-n),0)$	$a(t) = \sqrt{1-2n(n-1)}\,(t - t_0)$
$\mathcal{I}^*$	$\left(\frac{2(n-2)}{2n-1},\frac{(5-4n)n}{2n^2-3n+1},\frac{5-4n}{2n^2-3n+1},0\right)$	$a(t) = a_0\,(t - t_0)^{\frac{2n^2-3n+1}{2-n}}$
$\mathcal{L}$	$\left(-\frac{3(n-1)(w+1)}{n},\frac{-4n+3w+3}{2n},\frac{-4n+3w+3}{2n^2},\frac{-2(3w+4)n^2+(9w+13)n-3(w+1)}{2n^2}\right)$	$a(t) = a_0\,(t - t_0)^{\frac{2n}{3(w+1)}}$

The Matter Power Spectrum (S=1,n>1,w=0)





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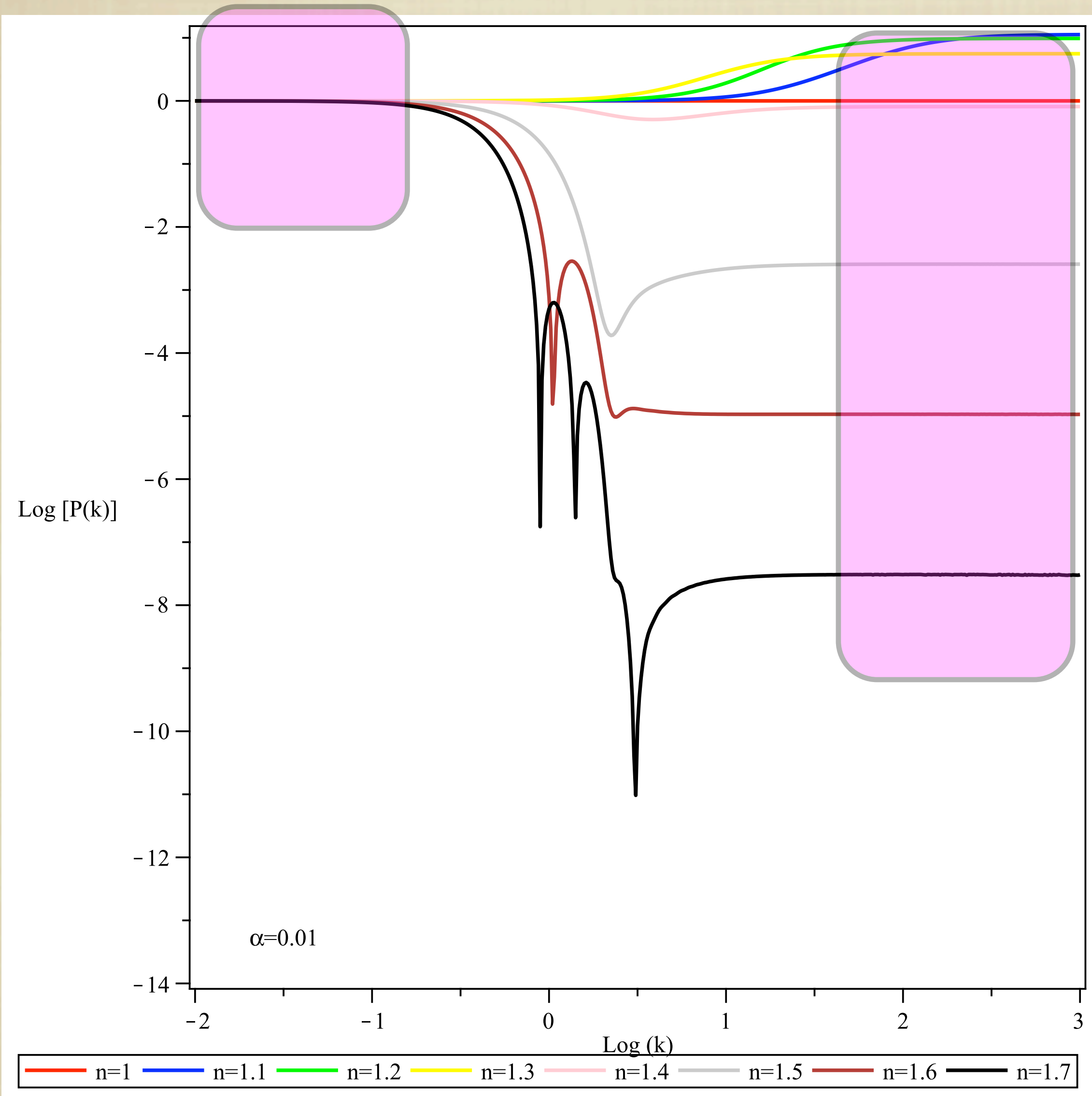

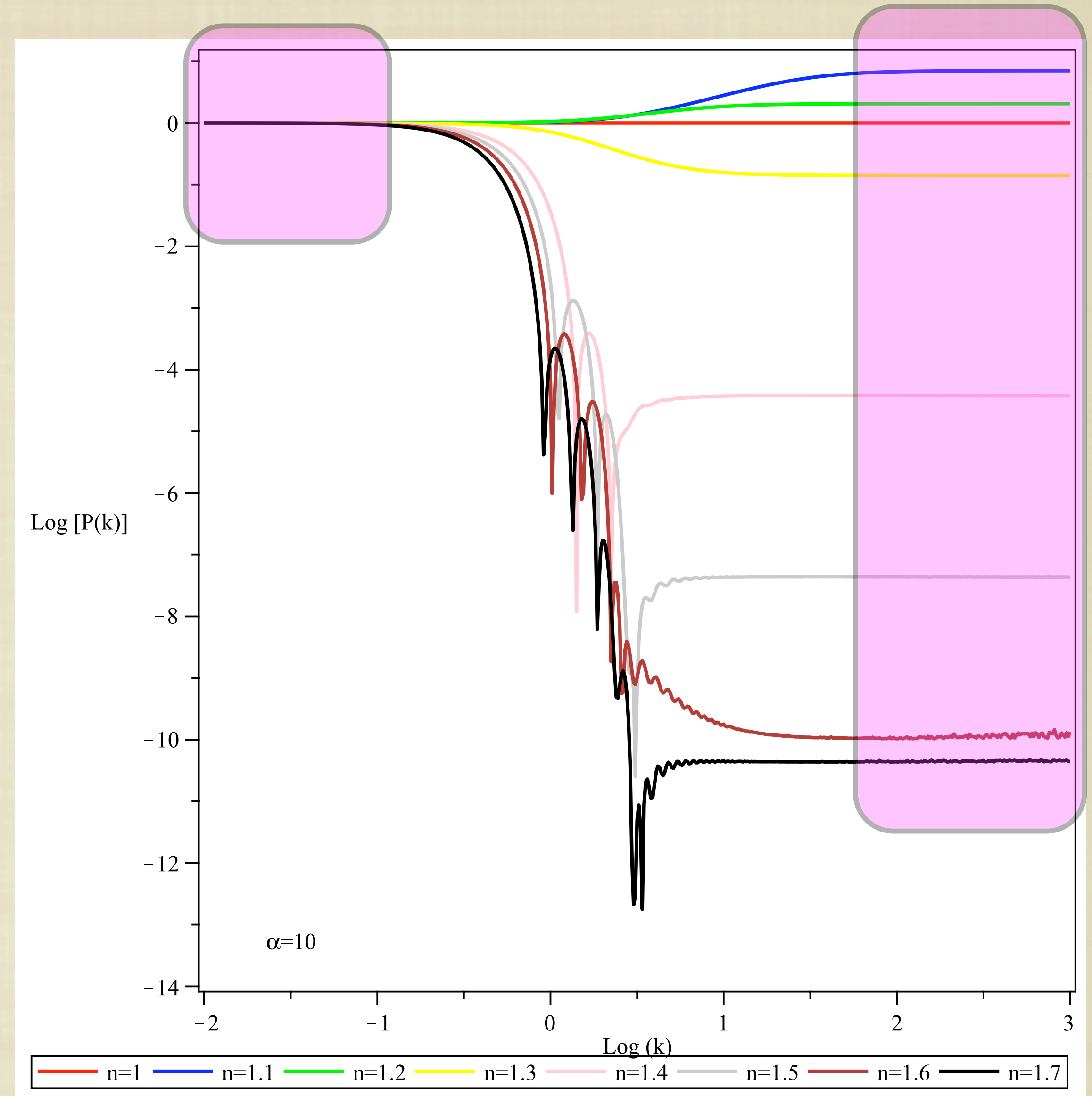

Same features of the previous example are independent of the values of α i.e. the value of the coupling only affects the dynamics

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WORK IN PROGRESS with other models.
PROBLEM: we don't really know much about their background.

IF verified, this result would constitute a clear and relatively easy way to probe fourth order gravity on cosmological scale.

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Conclusions

- We have used the covariant approach to investigate the behavior of scalar perturbations for a generic fourth gravity theory

- We have analyzed in detail some examples gaining a deeper understanding of the features of the matter era in this framework .

- There is strong indication that the spectrum of the scalar perturbations in f(R)-gravity presents a characteristic signature which could be a crucial test of the validity of these schemes.

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