The evolution of cosmological perturbations in higher order theories of gravity

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Projects underway

- A detailed analysis of the dynamics of f(R) gravity.
  - Locate special exact solutions (fixed points)
  - Determine how these fixed points relate to each other.
  - For a given f(R) theory of gravity, determine the region of phase-space compatible with observational constraints.
- The evolution of perturbations in f(R) gravity
  - Structure formation and gravitational waves,
  - Newtonian perturbations
  - Gravitational Lensing,
  - Growth function and CMB Anisotropies.

With Sante Carloni, Antonio Troisi, Salvatore Capozziello, Kishore Ananda , Jannie Leach and M. Abdelwahab.

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1+3 covariant approach

* From the time-like flow u^a we construct the projection onto surfaces orthogonal to the flow: $h_{ab} = g_{ab} + u_a u_b$.
* Covariant convective derivative on scalar: $\dot{f} = u^a \nabla_a f$.
* Spatial covariant derivative: $\tilde{\nabla}_a f = h^b{}_a \nabla_b f$.
* Kinematics of u^a gives geometry of congruence of flow lines.

$$\nabla_a u_b = -\overbrace{u_a \dot{u}_b}^{\text{acceleration}} + \overbrace{\tfrac{1}{3}\Theta h_{ab}}^{\text{expansion}} + \overbrace{\sigma_{ab}}^{\text{shear}} + \overbrace{\omega_{ab}}^{\text{vorticity}}$$

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a - \tilde{\nabla}^a \dot{u}_a + \dot{u}_a \dot{u}^a + \tfrac{1}{2}(\mu^{tot} + 3p^{tot}) = 0$$

Raychaudhuri equation.

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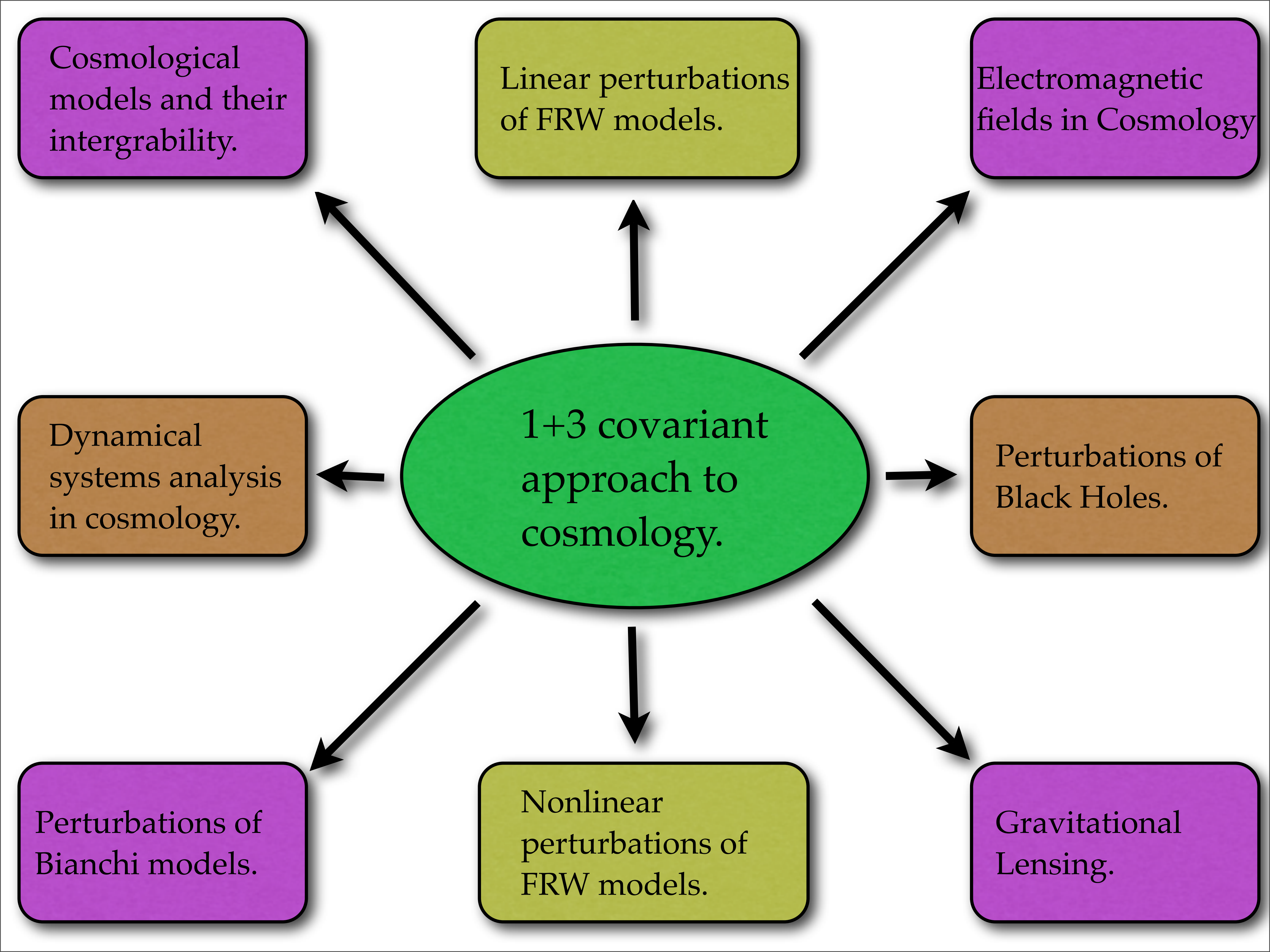

Cosmological models and their intergrability.
Linear perturbations of FRW models.
Electromagnetic fields in Cosmology
Dynamical systems analysis in cosmology.
1+3 covariant approach to cosmology.
Perturbations of Black Holes.
Perturbations of Bianchi models.
Nonlinear perturbations of FRW models.
Gravitational Lensing.

Fourth order gravity

The class of models we will consider can be derived from the classical action:

$$\boxed{\mathcal{A} = \int d^4x\sqrt{-g}\left[f(R) + \mathcal{L}_m\right]}\,,$$

Varying the action with respect to the metric gives the following field equations:

$$f'G_{ab} = f'\left(R_{ab} - \frac{1}{2}\,g_{ab}R\right) = T^m_{ab} + \frac{1}{2}g_{ab}\left(R - Rf'\right) + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f'\,,$$

$$\boxed{G_{ab} = \tilde{T}^m_{ab} + T^R_{ab} = T^{tot}_{ab}\,,}$$

This last step is extremely important as it allows us to treat 4th order gravity as standard GR in the presence of two effective fluids. It is this that makes using the covariant approach particularly straightforward.

The energy-momentum tensor of the curvature "fluid" can be decomposed as follows:

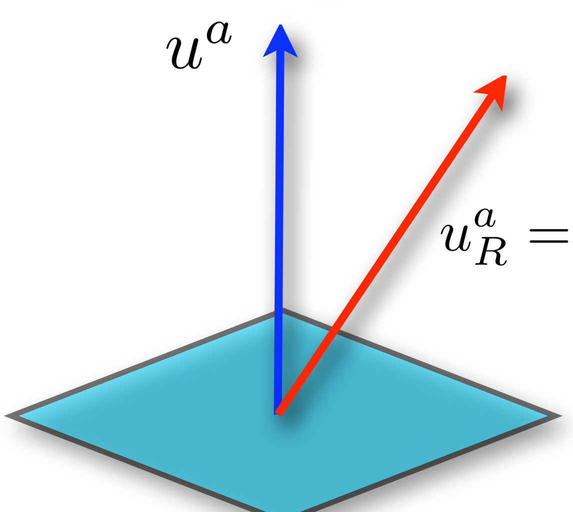
$$\mu^R = \frac{1}{f'}\left[\frac{1}{2}(Rf'-f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R + f''\,\dot{u}_b\tilde{\nabla} R\right] ,$$

$$p^R = \frac{1}{f'}\left[\frac{1}{2}(f-Rf') + f''\ddot{R} + 3f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R} - \frac{2}{3}f''\tilde{\nabla}^2 R + \right.$$
$$\left. -\frac{2}{3}f'''\tilde{\nabla}^a R\tilde{\nabla}_a R - \frac{1}{3}f''\,\dot{u}_b\tilde{\nabla} R\right] ,$$

Note no background contribution.

$$q_a^R = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\tilde{\nabla}_a R\right] ,$$

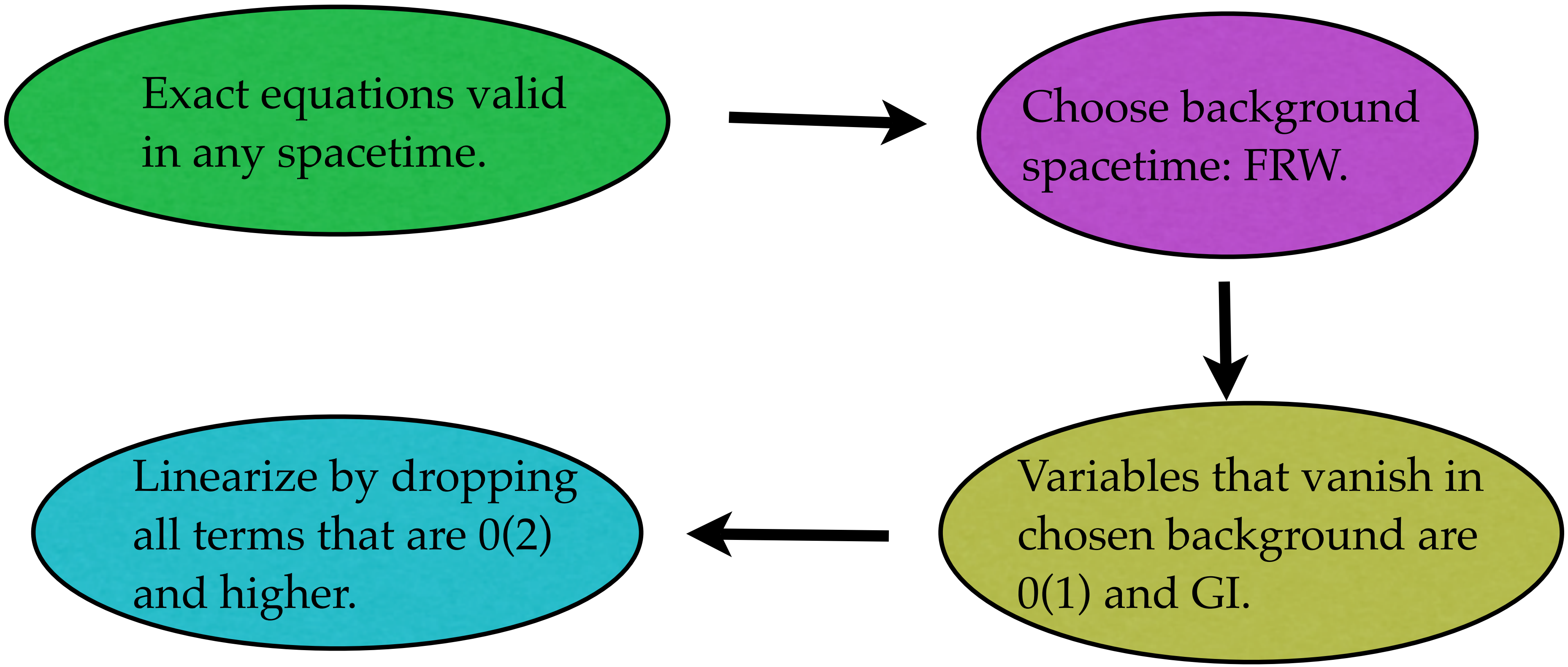
$$\pi_{ab}^R = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R + \sigma_{ab}\dot{R}\right] .$$



$$u_R^a = -\frac{\nabla_a R}{\dot{R}}$$

So one can think of this as a curvature "fluid" moving relative to u^a

Linearisation



$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \boxed{\sigma_{ab}\sigma^{ab}} - \boxed{2\omega_a\omega^a} - \tilde{\nabla}^a\dot{u}_a + \boxed{\dot{u}_a\dot{u}^a} + \frac{1}{2}(\mu^{tot} + 3p^{tot}) = 0$$

$$\dot{\Theta} + \frac{1}{3}\Theta^2 - \tilde{\nabla}^a\dot{u}_a + \frac{1}{2}(\mu^{tot} + 3p^{tot}) = 0 \longrightarrow \text{Almost FRW model.}$$

The linear gravitational equations

Propagation

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - \tilde{\nabla}^a A_a + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) = \boxed{-\tfrac{1}{2}(\mu^R + 3p^R)}\,,$$

$$\dot{\omega}_a + 2H\omega_a + \tfrac{1}{2}\mathrm{curl}\, A_a = 0\,,$$

$$\dot{\sigma}_{ab} + 2H\sigma_{ab} + E_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} = \boxed{-q_a^R}\,,$$

$$\dot{E}_{ab} + 3HE_{ab} - \mathrm{curl}\, H_{ab} + \tfrac{1}{2}(\tilde{\mu}^m + \tilde{p}^m)\sigma_{ab} = \boxed{-\tfrac{1}{2}(\mu^R + p^R)\sigma_{ab} - \tfrac{1}{2}\dot{\pi}^R_{\langle ab\rangle} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle} - \tfrac{1}{6}\Theta\pi^R_{ab}}\,,$$

$$\dot{H}_{ab} + 3HH_{ab} + \mathrm{curl}\, E_{ab} = \boxed{\tfrac{1}{2}\mathrm{curl}\, \pi^R_{ab}\,,}$$

Constraint

$$\tilde{\nabla}^b\sigma_{ab} - \mathrm{curl}\,\omega_a - \tfrac{2}{3}\tilde{\nabla}_a\Theta = \boxed{-q_a^R}\,,$$

$$\mathrm{curl}\,\sigma_{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} = 0\,,$$

$$\tilde{\nabla}^b E_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\tilde{\mu}^m = \boxed{-\tfrac{1}{2}\tilde{\nabla}^b\pi^R_{ab} + \tfrac{1}{3}\tilde{\nabla}_a\mu^R - \tfrac{1}{3}\Theta q_a^R}\,,$$

$$\tilde{\nabla}^b H_{ab} - (\tilde{\mu}^m + \tilde{p}^m)\omega_a = \boxed{-\tfrac{1}{2}\mathrm{curl}\, q_a^R + (\mu^R + p^R)\omega_a}\,,$$

$$\tilde{\nabla}^a\omega_a = 0\,,$$

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The linear conservation equations

The Bianchi identities:
$$\left\{\begin{aligned}
\tilde{T}^{M;b}_{ab} &= \frac{T^{m;b}_{ab}}{f'} - \frac{f''}{f'^2}\, T^{m}_{ab}\; R^{;b}\;,\\
T^{R;b}_{ab} &= \frac{f''}{f'^2}\,\tilde{T}^{M}_{ab}\; R^{;b}\;,
\end{aligned}\right.$$

Matter
$$\left\{\begin{aligned}
\dot{\mu}^m \; &= \; -\,\Theta\,(\mu^m + p^m)\;,\\
\tilde{\nabla}^a p^m &= -(\mu^m + p^m)\,\dot{u}^a\;,
\end{aligned}\right.$$

Curvature
$$\left\{\begin{aligned}
&\dot{\mu}^R + \tilde{\nabla}^a q^R_a = -\,\Theta\,(\mu^R + p^R) + \boxed{\mu^m\,\frac{f''\,\dot{R}}{f'^2}\;,}\\
&\dot{q}^R_{\langle a\rangle} + \tilde{\nabla}_a p^R + \tilde{\nabla}^b \pi^R_{ab} = -\,\tfrac{4}{3}\,\Theta\, q^R_a - (\mu^R + p^R)\,\dot{u}_a + \boxed{\mu^m\,\frac{f''\,\tilde{\nabla}_a R}{f'^2}}\,,
\end{aligned}\right.$$

The background equations

Friedmann $\left\{ H^2 + \frac{k}{S^2} = \frac{1}{3f'} \left\{ \frac{1}{2}\left[f'R - f(R)\right] - 3H\dot{f}' + \mu_m \right\} \right.$,

Raychaudhuri $\left\{ 2\dot{H} + H^2 + \frac{k}{S^2} = -\frac{1}{f'} \left\{ \frac{1}{2}\left[f'R - f(R)\right] + \ddot{f}' - 3H\dot{f}' + p_m \right\} \right.$,

Conservation $\left\{ \dot{\mu}_m + 3H(\mu_m + p_m) = 0 \right.$,

$$R = -6\left(2H^2 + \dot{H} + \frac{k}{a^2}\right) ,$$

.......But how do we find useful background solutions?

The dynamical systems approach

$$x = \frac{\dot{f'}}{f'H}, \qquad y = \frac{R}{6H^2}, \qquad z = \frac{f}{6f'H^2},$$

$$\Omega = \frac{\mu_m}{3f'H^2}, \qquad K = \frac{k}{a^2H^2}\,, \quad \mathfrak{q} \equiv \left(\frac{d\log F}{d\log R}\right)^{-1} = \frac{f'}{Rf''}\,.$$

Autonomous set of equations

$$\begin{cases} \dfrac{dx}{dN} = \varepsilon\left(2K + 2z - x^2 + (K + y + 1)x\right) + \Omega\varepsilon\left(-3w - 1\right) + 2, \\ \dfrac{dy}{dN} = y\varepsilon\left(2y + 2K + x\mathfrak{q} + 4\right), \\ \dfrac{dz}{dN} = z\varepsilon\left(2K - x + 2y + 4\right) + x\varepsilon\, y\mathfrak{q}, \\ \dfrac{d\Omega}{dN} = \Omega\varepsilon\left(2K - x + 2y - 3w + 1\right), \\ \dfrac{dK}{dN} = K\varepsilon\left(2K + 2y + 2\right), \end{cases}$$

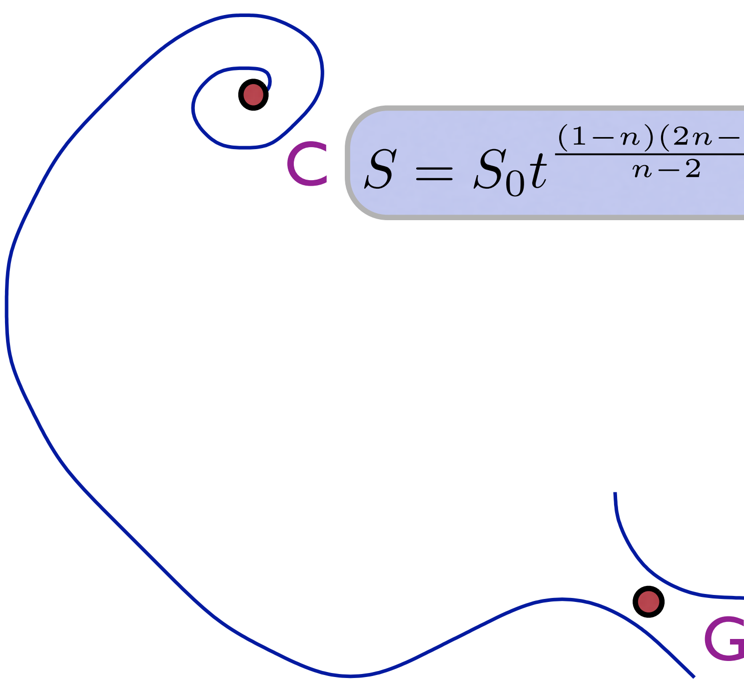
Constraint

$$1 = -K - x - y + z + \Omega\,,$$

- Carloni, Dunsby Capozziello, Troisi (CQG, 2005)
- Amendola et. al. (PRD, 2007)
- Carloni, Dunsby, Troisi (arXiv: 0706.0452)

A simple example: R^n gravity

Point	Coordinates (x, y, z)	Scale Factor
$\mathcal{A}$	$[0, 0, 0]$	$a = a_0(t - t_0)$
$\mathcal{B}$	$[-1, 0, 0]$	$a = a_0(t - t_0)^{1/2}$ (only for $n = 3/2$)
$\mathcal{C}$	$\left[\frac{2(n-2)}{2n-1}, \frac{4n-5}{2n-1}, 0\right]$	$a = a_0\, t^{\frac{(1-n)(2n-1)}{n-2}}$
$\mathcal{D}$	$[2(1 - n), 2(n - 1)^2, 0]$	$\begin{cases} a = \frac{kt}{2n^2-2n-1} & \text{if} \quad k \neq 0 \\ a = a_0 t & \text{if} \quad k = 0 \end{cases}$
$\mathcal{E}$	$[-1 - 3\omega, 0, -1 - 3\omega]$	$a = a_0(t - t_0)$
$\mathcal{F}$	$[1 - 3\omega, 0, 2 - 3\omega]$	$a = a_0(t - t_0)^{1/2}$ (only for $n = 3/2$)
$\mathcal{G}$	$\left[-\frac{3(n-1)(1+\omega)}{n}, \frac{(n-1)[4n-3(\omega+1)]}{2n^2}, \frac{n(13+9\omega)-2n^2(4+3\omega)-3(1+\omega)}{2n^2}\right]$	$a = a_0\, t^{\frac{2n}{3(1+\omega)}}$



$$S = S_0 t^{\frac{(1-n)(2n-1)}{n-2}}$$

1.36<n<1.5

$$S = S_0 t^{\frac{2n}{3(1+w)}}$$

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Background quantities

If we choose point G: $\boxed{S = S_0 t^{\frac{2n}{3(1+w)}}}$, $\qquad k = 0\,, \qquad \mu = \mu_0 t^{-2n}$

Then the background quantities can be easily calculated:

$$
\begin{aligned}
\Theta &= \frac{2n}{t(\omega+1)}\,, \\
R &= \frac{4n[4n-3(\omega+1)]}{3t^2(\omega+1)^2}\,, \\
\mu^R &= \frac{2(n-1)[2n(3\omega+5)-3(\omega+1)]}{3t^2(\omega+1)^2}\,, \\
p^R &= \frac{2(n-1)\left[n\left(6\omega^2+8\omega-2\right)-3\omega(\omega+1)\right]}{3t^2(\omega+1)^2}\,, \\
\mu &= \left(\frac{3}{4}\right)^{1-n} n\chi \left(\frac{n(4n-3(\omega+1))}{t^2(\omega+1)^2}\right)^{n-1} \frac{4n^2-2(n-1)[2n(3\omega+5)-3(\omega+1)]}{3(\omega+1)^2 t^2}\,.
\end{aligned}
$$

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Gravitational Waves

Linear gravitational waves are described by the transverse trace-free degrees of freedom once scalars and vectors have been switched off.

- The 1+3 covariant variables relevant for gravitational waves are:
  - The shear tensor
  - The electric part of the Weyl tensor
  - The Magnetic part of the Weyl tensor
- The evolution of these variables can be calculated in general, but we will focus on their evolution in an almost FRW spacetime.

The evolution of tensor perturbations are governed by three propagation equations:

$$\dot{\sigma}_{ab} + \frac{2}{3}\,\Theta\,\sigma_{ab} + E_{ab} - \boxed{\frac{1}{2}\pi_{ab}} = 0\ ,$$

$$\dot{H}_{ab} + H_{ab}\,\Theta + (\mathrm{curl}\,E)_{ab} - \boxed{\frac{1}{2}(\mathrm{curl}\,\pi)_{ab}} = 0\ ,$$

$$\dot{E}_{ab} + E_{ab}\,\Theta - (\mathrm{curl}\,H)_{ab} + \frac{1}{2}\,(\mu + p)\ \sigma_{ab} + \boxed{\frac{1}{6}\,\Theta\,\pi_{ab} + \frac{1}{2}\dot{\pi}_{ab}} = 0\ ,$$

and three constraints:

$$\tilde{\nabla}_b H^{ab} = 0\ , \quad \tilde{\nabla}_b E^{ab} = 0\ , \quad H_{ab} = (\mathrm{curl}\,\sigma)_{ab}\ .$$

Substituting for the "curvature fluid terms" and performing a tensor harmonic decomposition we obtain two wave equations for the shear and magnetic part of the Weyl tensor and a relation determining the electric part of the Weyl tensor.

$$\ddot\sigma^{(k)} + \left(\frac{5}{3}\Theta + \dot R\frac{f''}{f'}\right)\dot\sigma^{(k)} + \left\{\frac{1}{9}\Theta^2 + \frac{1}{f'}\left(\frac{1}{6}\mu^m - \frac{3}{2}p^m\right) + \frac{k^2}{a^2} - \frac{1}{2}\Theta\dot R\frac{f''}{f'} - \frac{5}{6}\frac{1}{f'}(f-f'R) - \dot R^2\left[\frac{1}{2}\frac{f'''}{f'} + \left(\frac{f''}{f'}\right)^2\right] - \frac{1}{2}\ddot R\frac{f''}{f'}\right\}\sigma^{(k)} = 0,$$

$$\ddot H^{(k)} + \left(\frac{7}{3}\Theta + \dot R\frac{f''}{f'}\right)\dot H^{(k)} + \left\{\frac{2}{3}\Theta^2 - \frac{2}{f'}p^m + \frac{k^2}{a^2} - \frac{1}{3}\Theta\dot R\frac{f''}{f'} - \frac{1}{f'}(f-f'R) - \dot R^2\left[\frac{f'''}{f'} + \left(\frac{f''}{f'}\right)^2\right] - \ddot R\frac{f''}{f'}\right\}H^{(k)} = 0,$$

$$E^{(k)} = -\dot\sigma^{(k)} - \left(\frac{2}{3}\Theta + \frac{1}{2}\dot R\frac{f''}{f'}\right)\sigma^{(k)}.$$

We investigate the following cases:

- The mater dominated era (w=0) relevant to direct detectors.
- The radiation dominated era (w=1/3) which is relevant to CMB constraints.

The physically relevant quantity is the dimensionless expansion normalized sheer:

$$\boxed{\Sigma^{(k)} = \frac{\sigma^{(k)}}{H}.}$$

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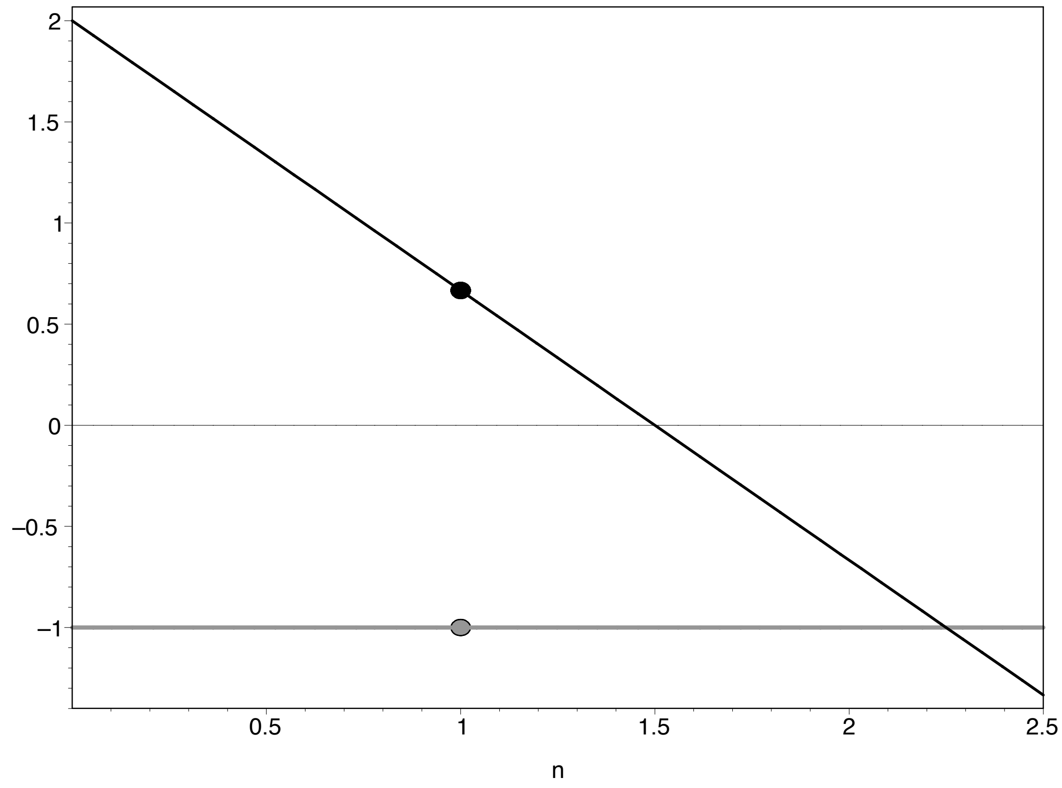
Matter dominated era

In the case of dust, the long wavelength solution is

$$\Sigma^{(k)} = \tilde{\Sigma}_1 t^{-1} + \tilde{\Sigma}_2 t^{2(1-r)}, \quad r = \frac{2n}{3}.$$



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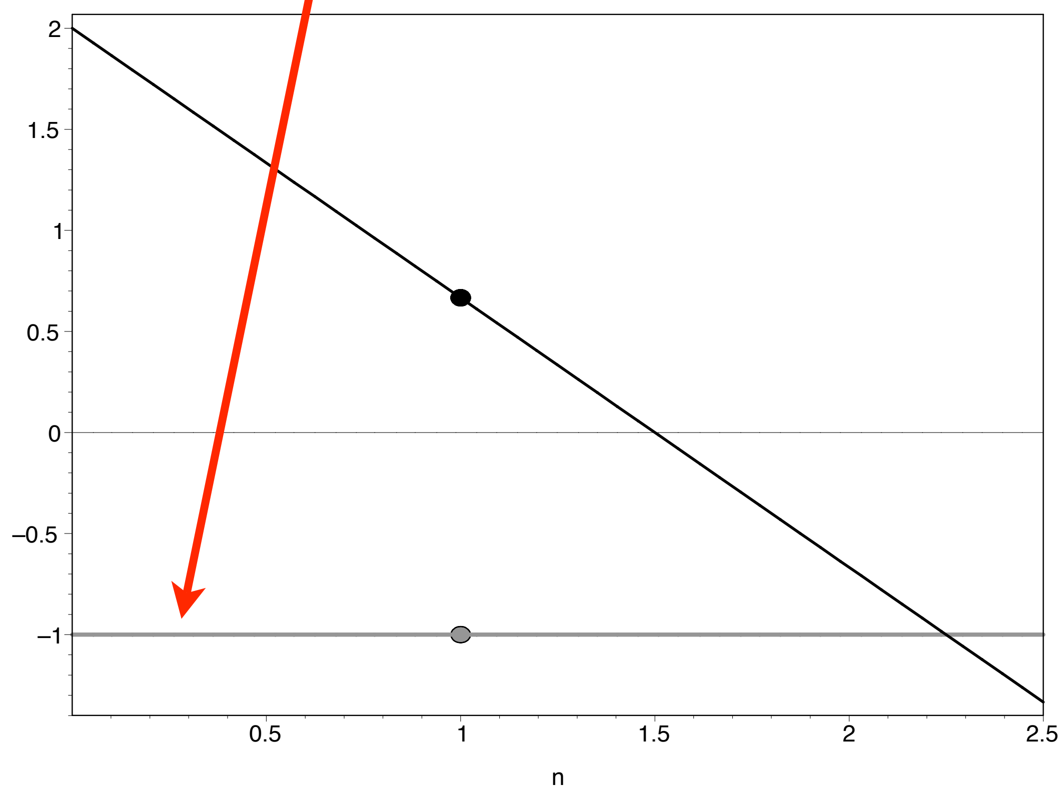

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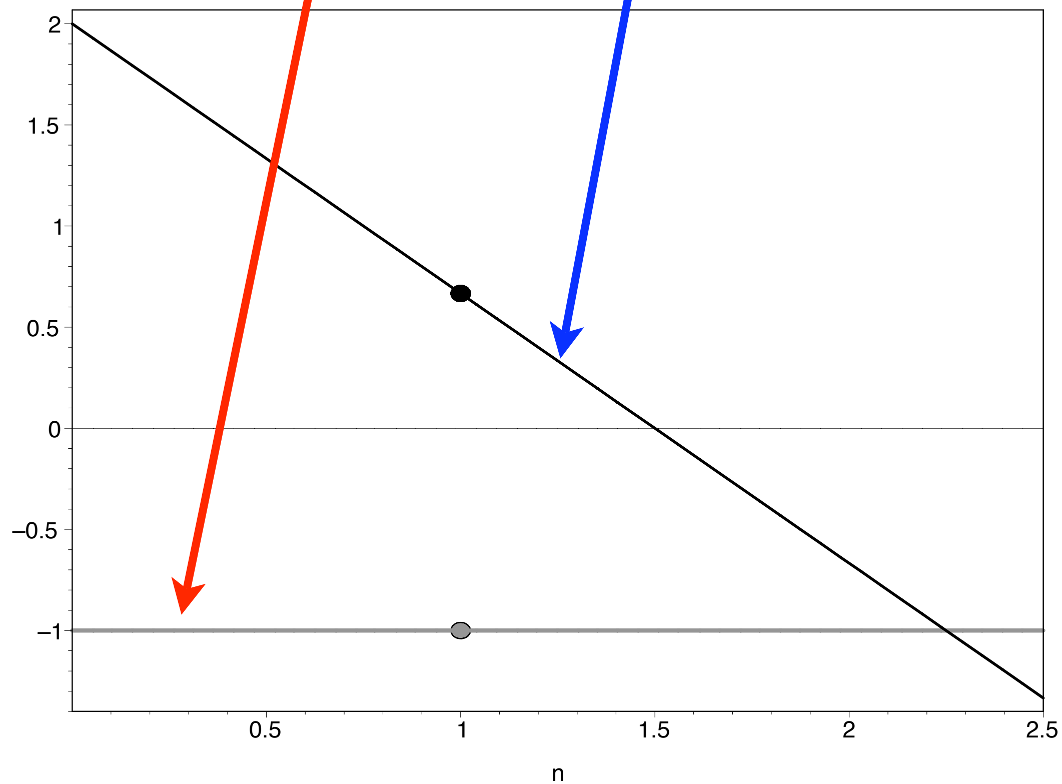

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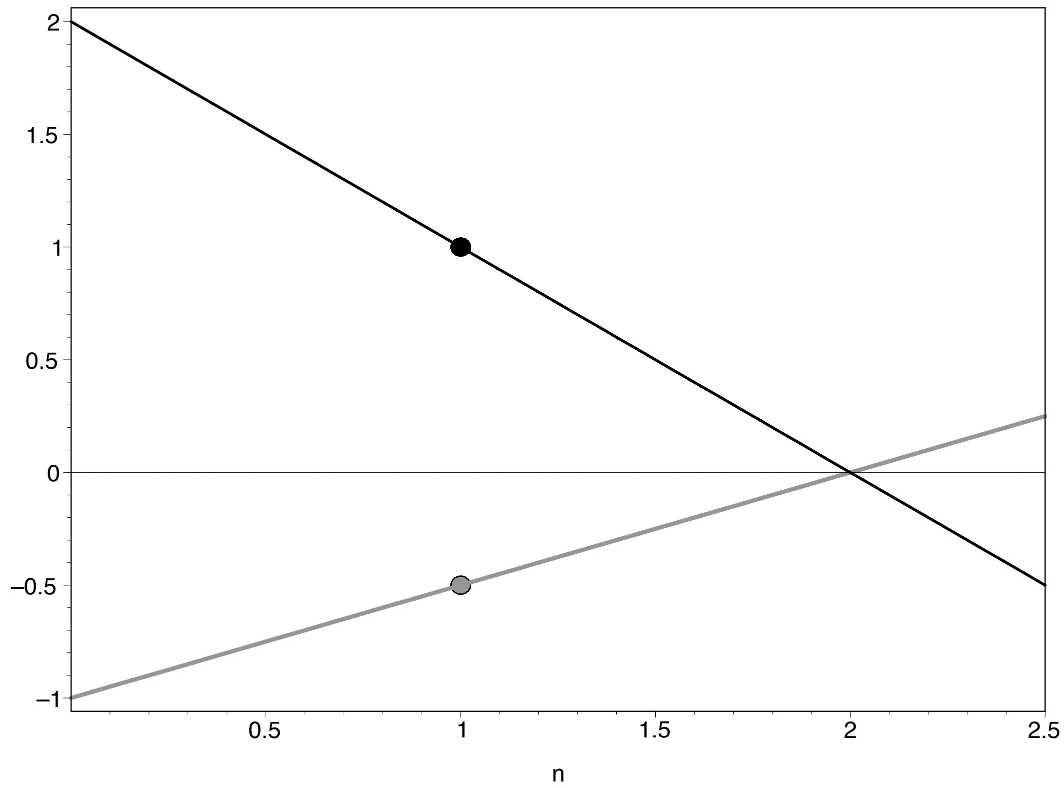
Radiation dominated era

In the case of radiation, the long wavelength solution is

$$\Sigma^{(k)} \;=\; \tilde{\Sigma}_1 t^{(2-2r)} + \tilde{\Sigma}_2 t^{(r-1)}, \quad r = \frac{n}{2}.$$



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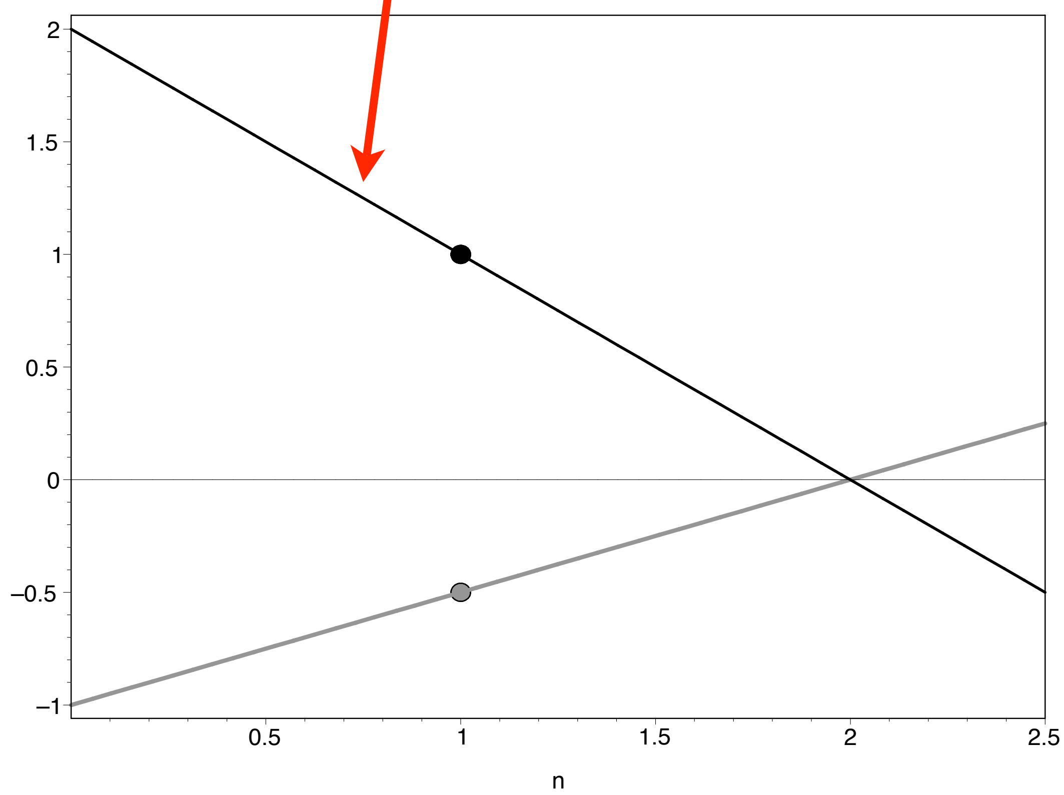

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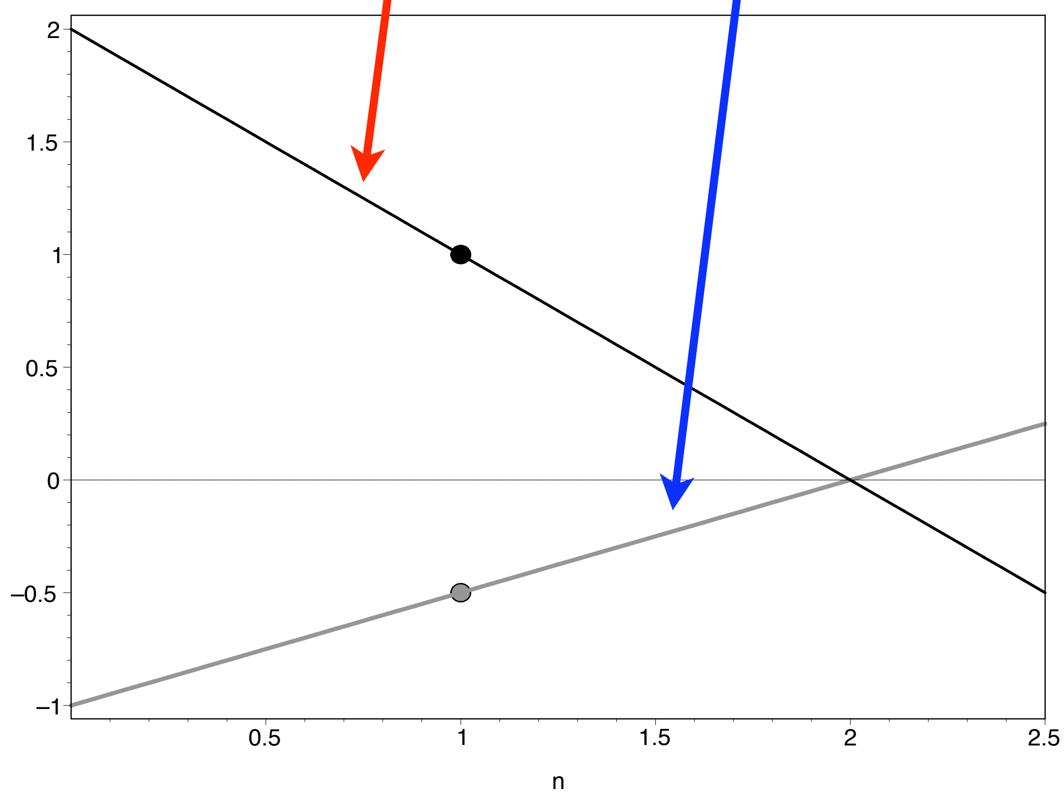
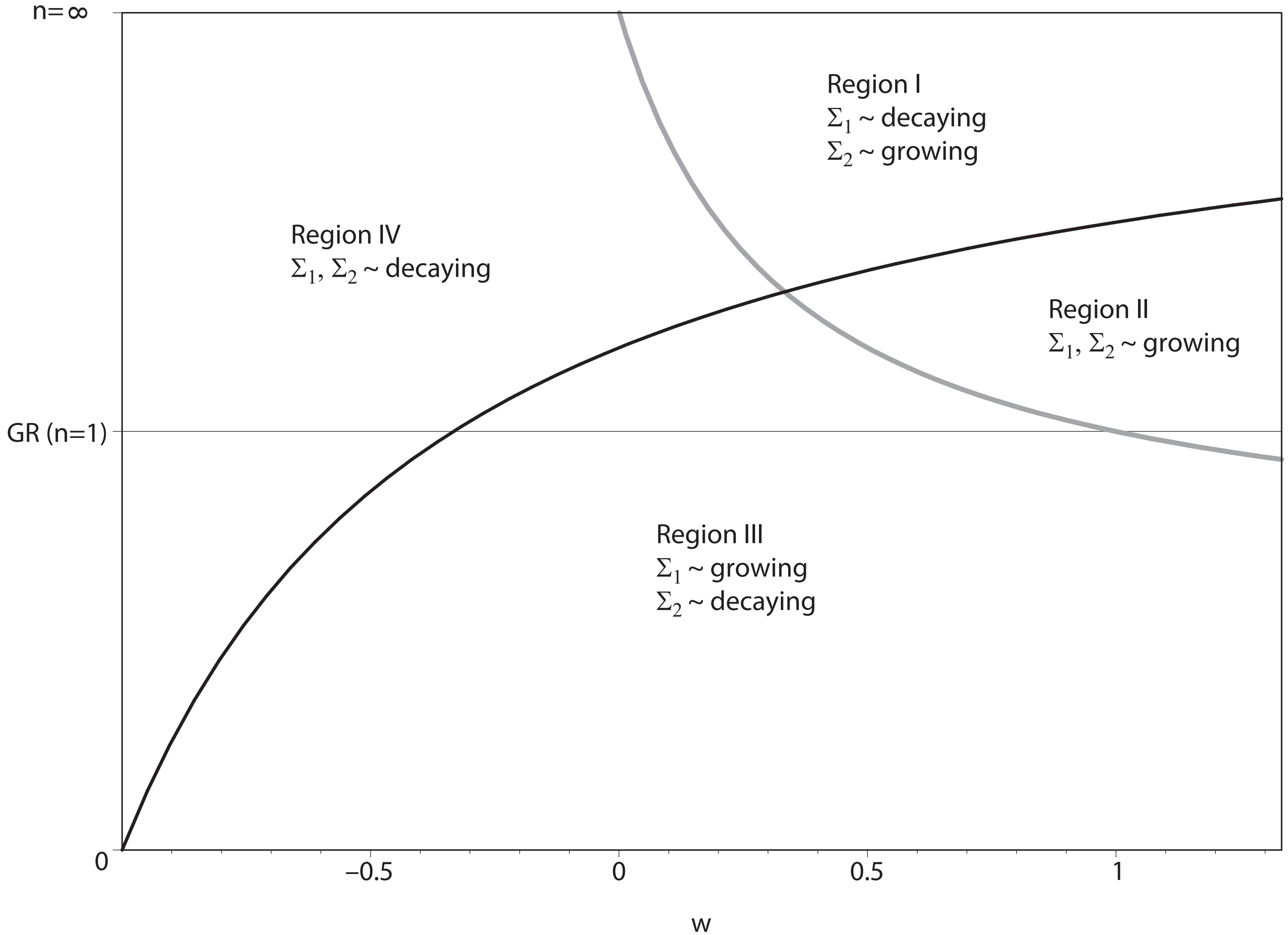


General equation of state

In general, the long wavelength solution is

$$\Sigma^{(k)} = \tilde{\Sigma}_1 t^{(2-2r)} + \tilde{\Sigma}_2 t^{(2n-1-3r)}. \quad r = \frac{2n}{3\left(1 + w_m\right)}.$$



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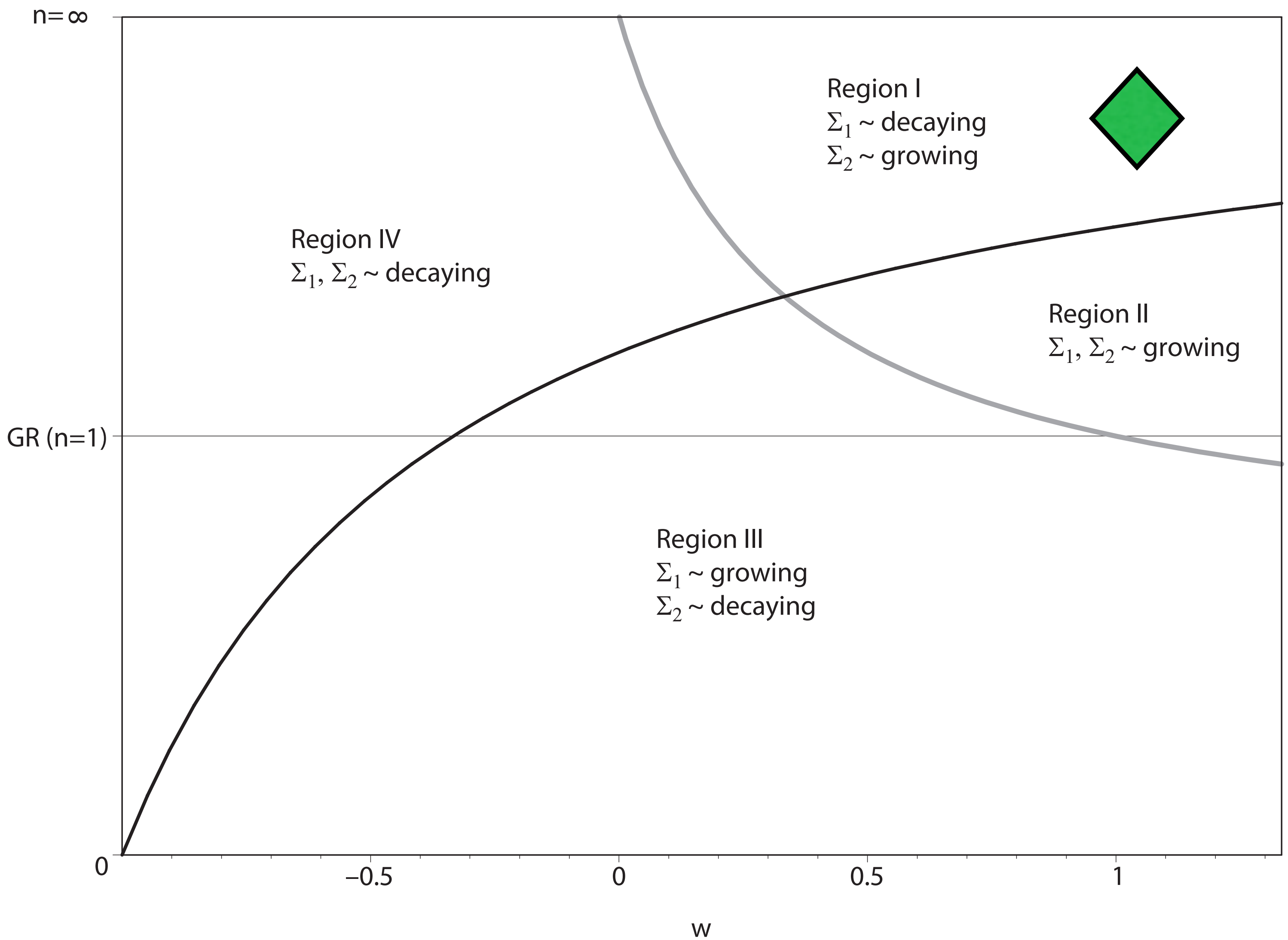

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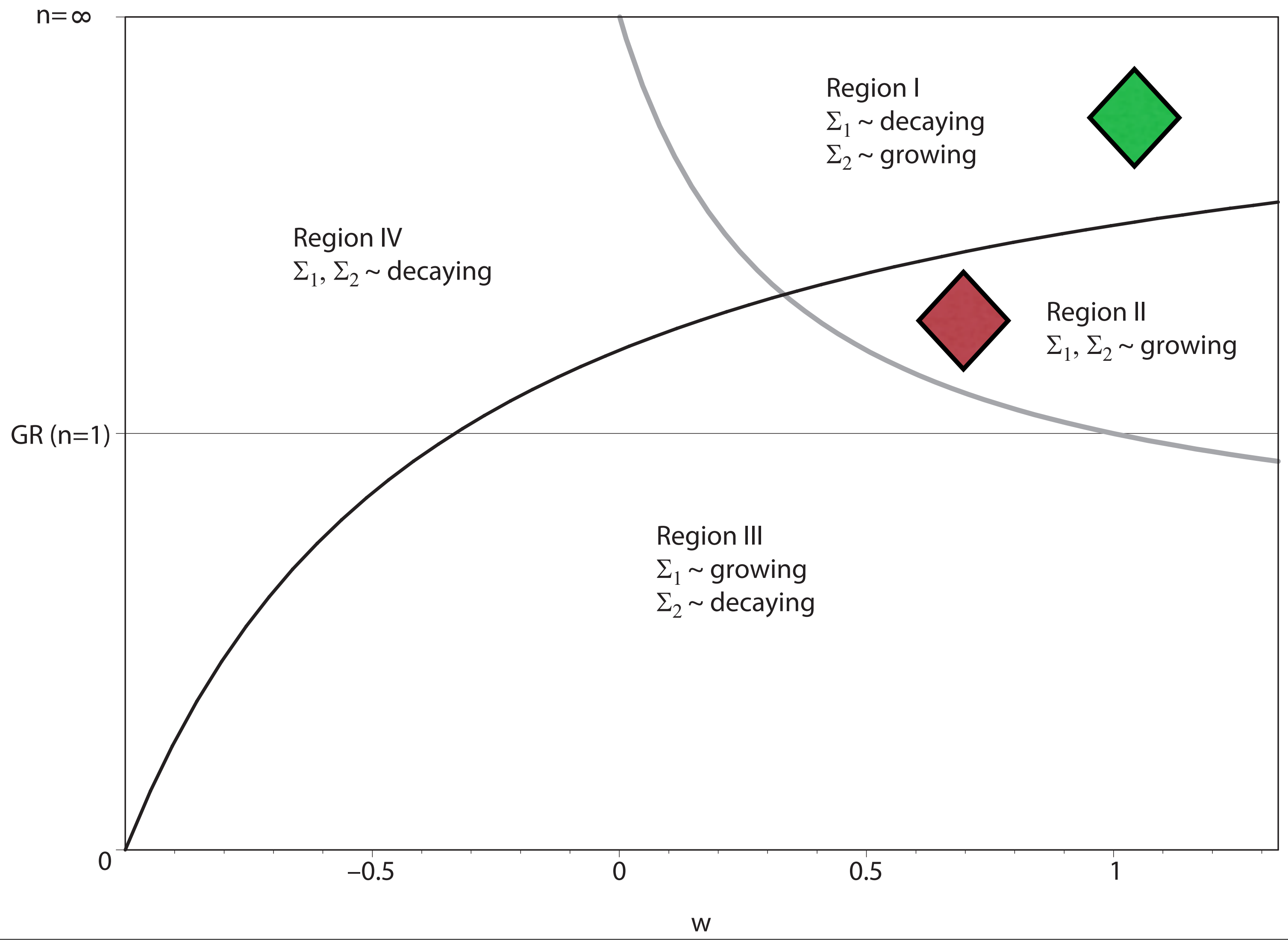

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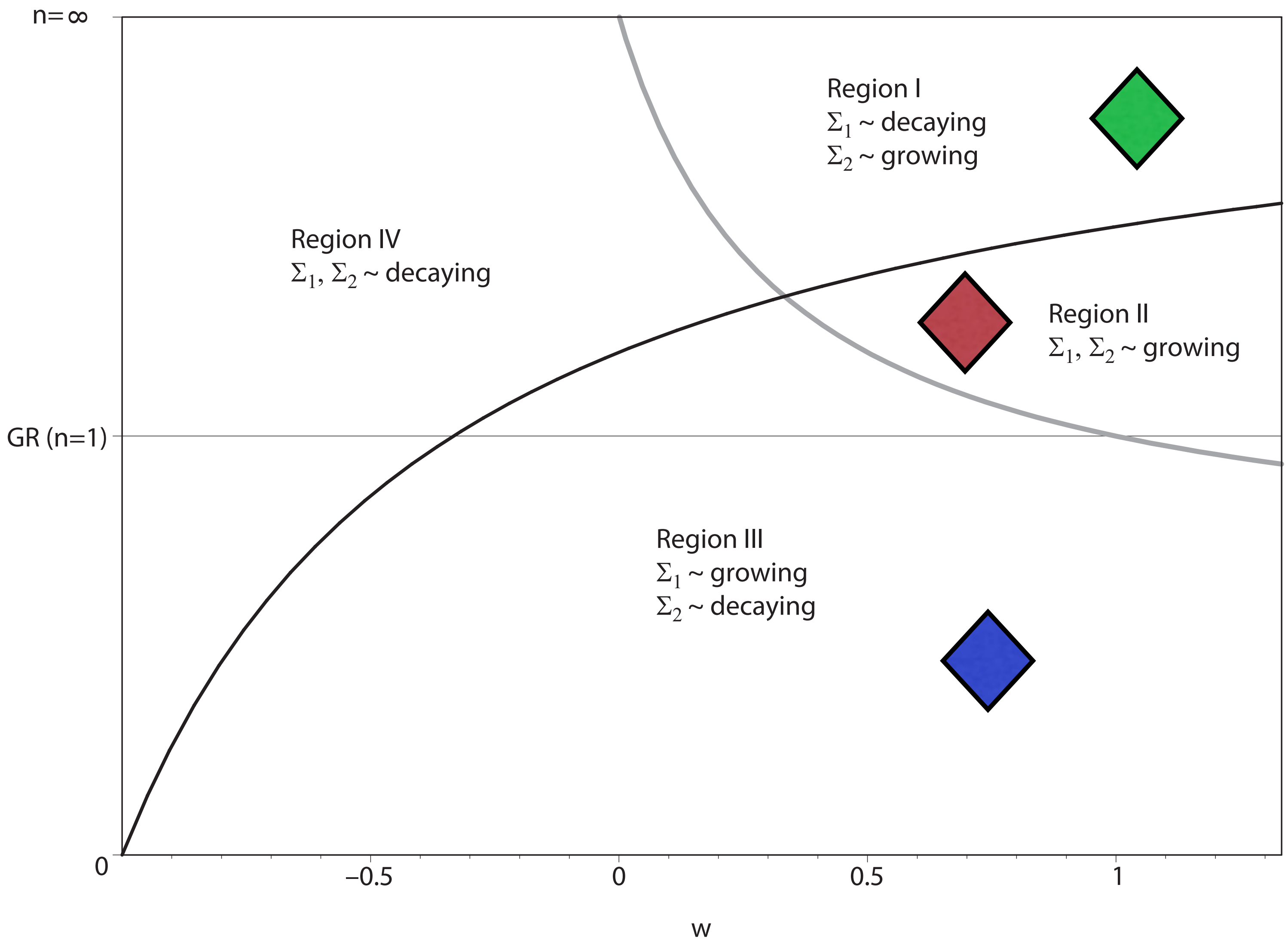

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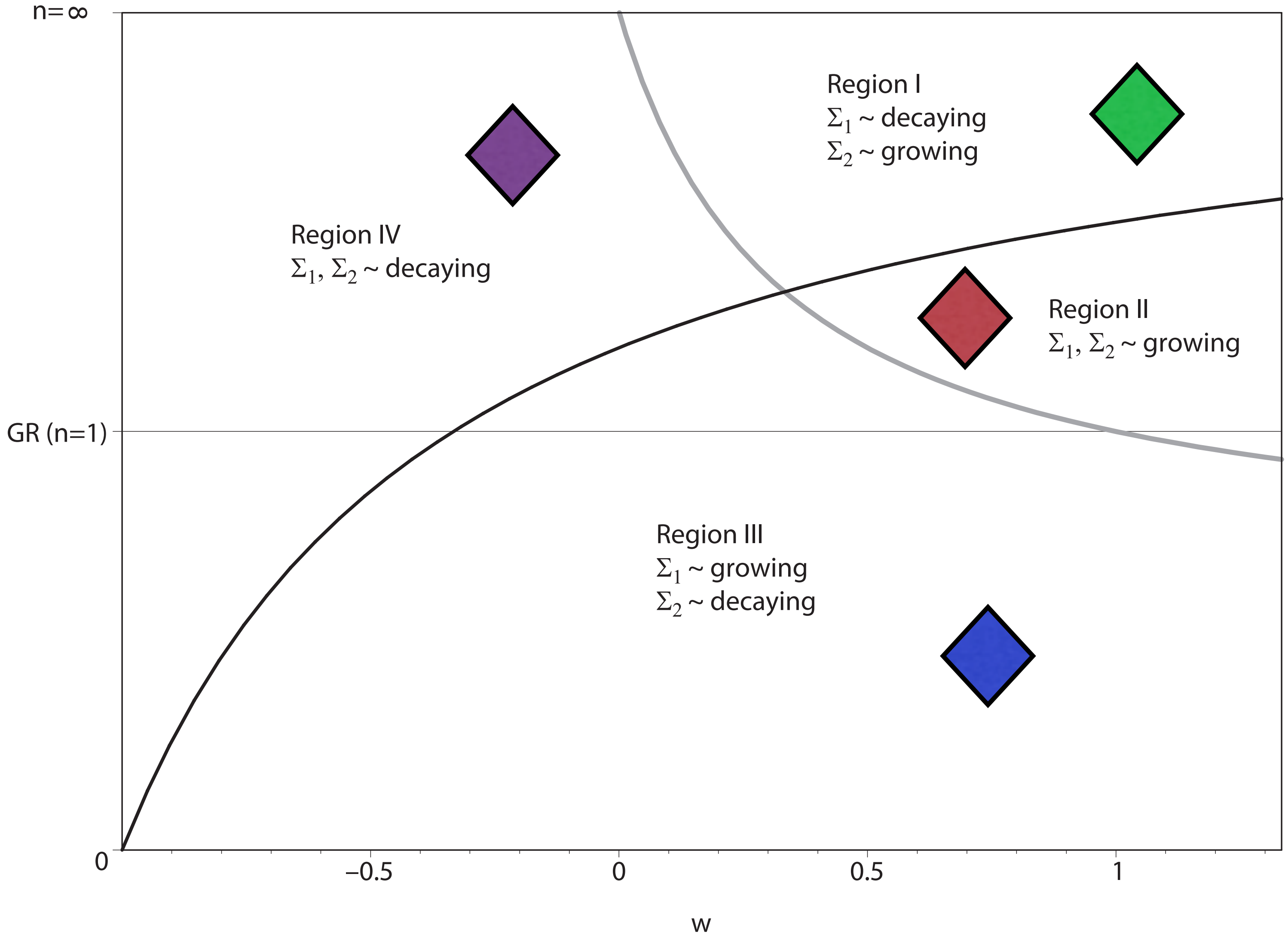

Relation to the Bardeen method

Expanding in w.r.t. the Bardeen metric gives expressions for the Covariant variables in terms of the TT metric perturbation.

$$\sigma_{\alpha\beta} = aH_T^{(2)\prime}Y^{(2)}_{\alpha\beta},$$

$$^{(3)}\mathcal{R}_{\alpha\beta} = \left(k^2+2K\right)H_T^{(2)}Y^{(2)}_{\alpha\beta},$$

$$E_{\alpha\beta} = -\frac{1}{2}\left[H_T^{(2)\prime\prime}-\left(k^2+2K\right)H_T^{(2)}\right]Y^{(2)}_{\alpha\beta},$$

$$H_{\alpha\beta} = a^{-2}H_T^{(2)\prime}Y^{(2)\gamma|\delta}_{(\alpha}\eta_{\beta)0\gamma\delta}.$$

Substituting these into the expression for the trace free part of the 3-Ricci tensor:

$$ {}^{(3)}\mathcal{R}_{\alpha\beta} \;=\; -\frac{\Theta}{3}\left(\sigma_{\alpha\beta}+\omega_{\alpha\beta}\right)+E_{\alpha\beta}+\frac{1}{2}\pi_{\alpha\beta}, $$

Gives the standard Bardeen tensor perturbation equation:

$$ H_T^{(2)\prime\prime}+2\frac{a'}{a}H_T^{(2)\prime}+\left(k^2+2K\right)H_T^{(2)} \;=\; \pi, $$

and putting the "curvature fluid" in gives:

$$ H_T^{(2)\prime\prime}+\left[2\frac{a'}{a}+\frac{\partial^2 f}{\partial R^2}\left(\frac{\partial f}{\partial R}\right)^{-1}R'\right]H_T^{(2)\prime}+\left(k^2+2K\right)H_T^{(2)} \;=\; 0, $$

Conclusion

* Equations for tensor perturbations were derived and solved for the matter and radiation eras in the long wavelength limit.

* Small departures from GR give significant modifications to GWs, so in principle GWs can provide strong constraints to FOG theories.

* Work is currently in progress to calculate the growth function and the CMB B-mode tensor anisotropies. WATCH THIS SPACE.

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