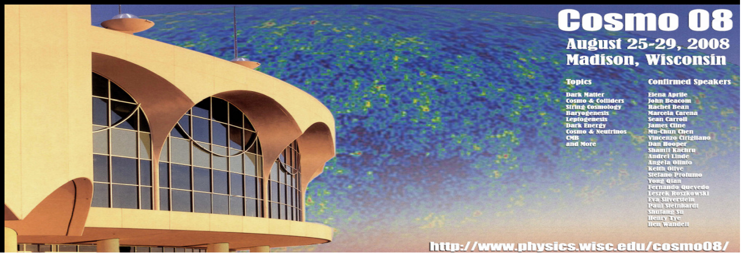
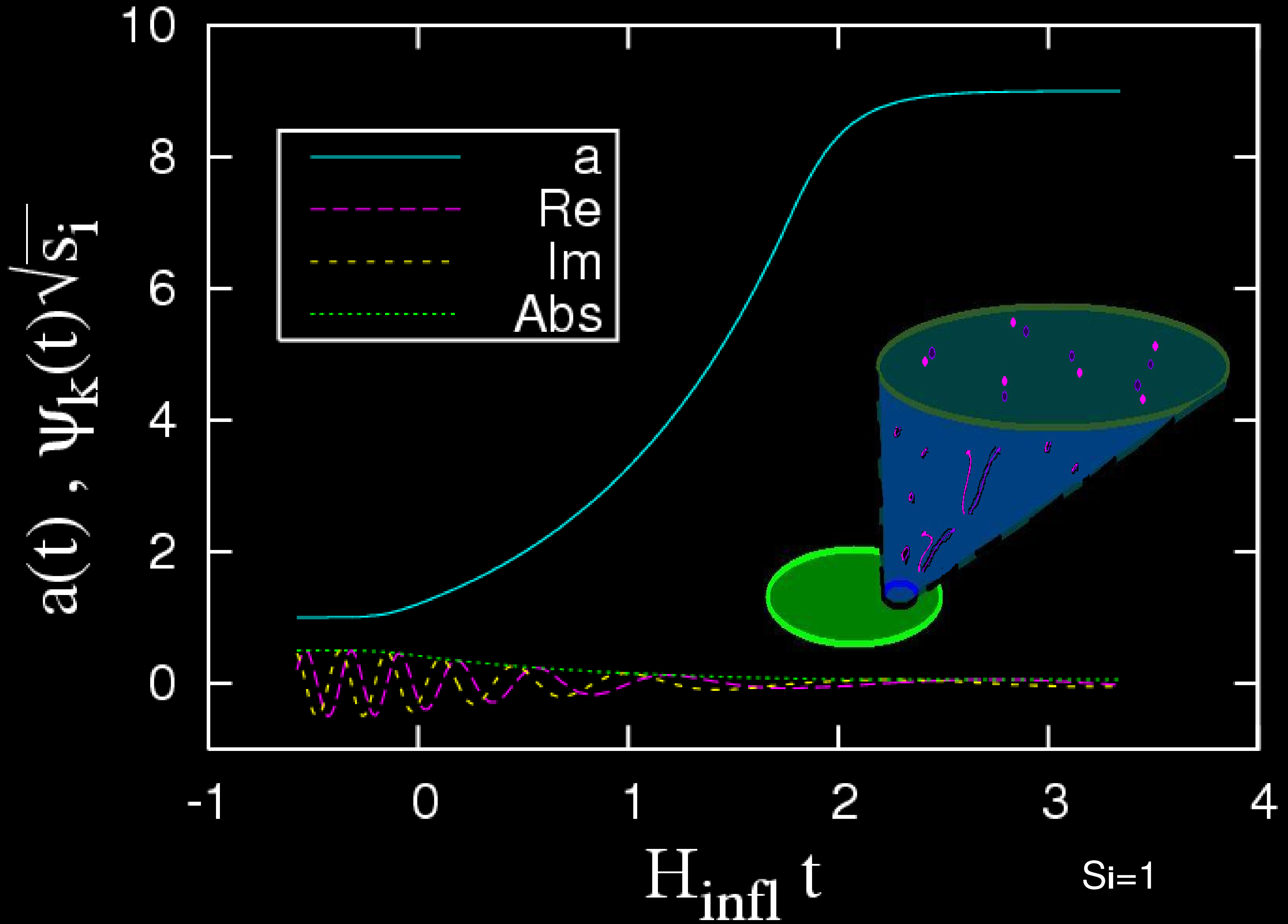
Dispersion Spectrum of Inflaton Perturbations Calculated Numerically with Reheating

Matthew Glenz

Research done with Leonard Parker,
Supported by National Space Grant College and Fellowship Program and the Wisconsin Space Grant Consortium; and by the Lynde and Harry Bradley Foundation.

Thursday, August 28th, 2008




10
8
6
4
2
0
-1
0
1
2
3
4
a
Re
Im
Abs
$a(t)\ ,\ \psi_k(t)\sqrt{s_i}$
$H_{infl}\ t$
Si=1

DEFINITIONS:

$$ds^2 = dt^2 - a^2(t)\,(dx^2 + dy^2 + dz^2)$$

$$\hbar = c = 1 \qquad a_{\text{infl}}(t) \propto e^{H_{\text{infl}} t}$$

$$H(t) \equiv \frac{da(t)/dt}{a(t)} \qquad q_2 \equiv \frac{k}{a_{\text{final}} H_{\text{infl}}}$$

$$m_H \equiv \frac{m}{H_{\text{infl}}} \qquad N_e \equiv \int H\, dt$$

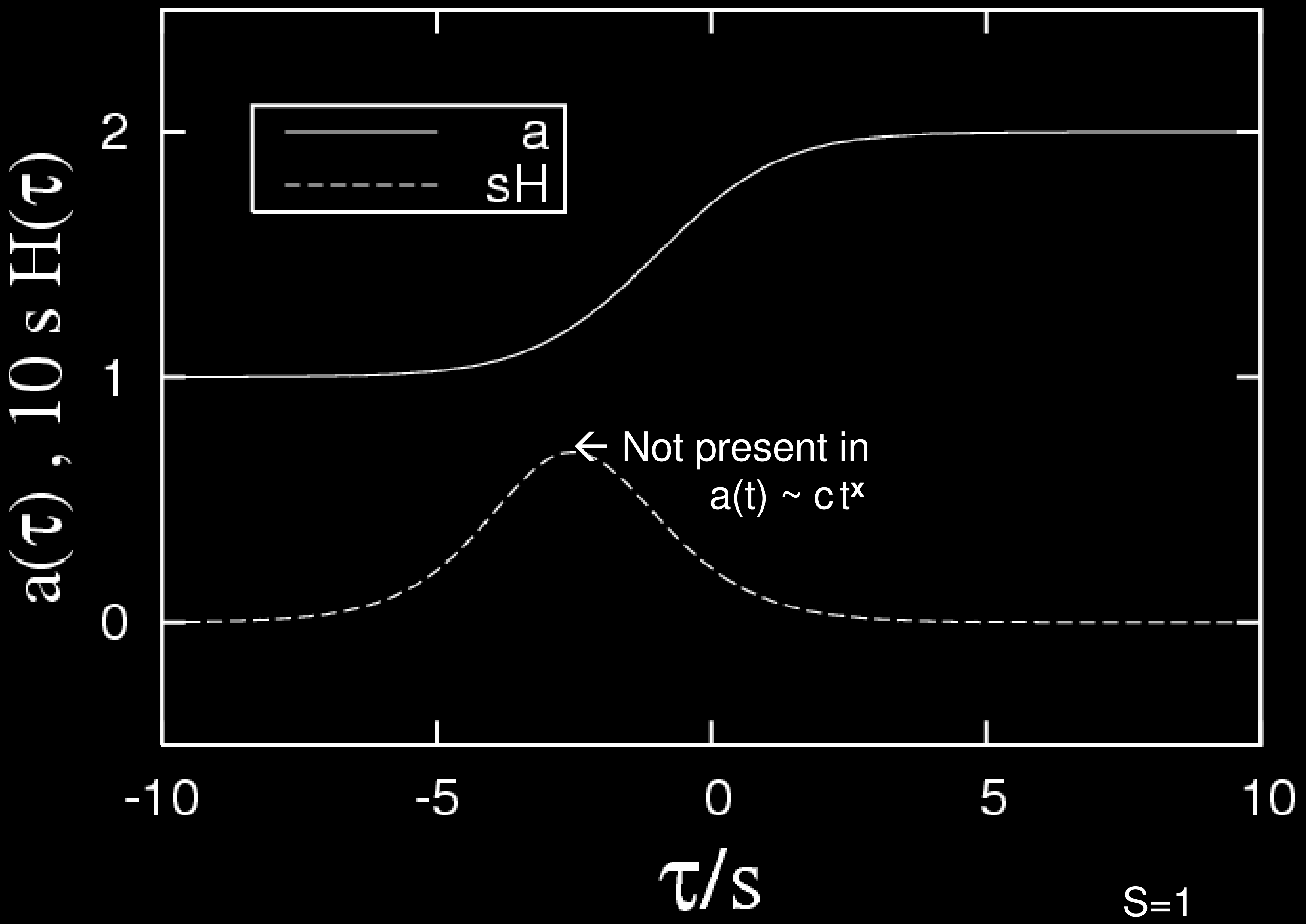
Massless case: exact.
m=const case: approximate.

Scalar field is asymptotic to Minkowski vacuum at early times.
(No infrared cutoff frequency necessary.)

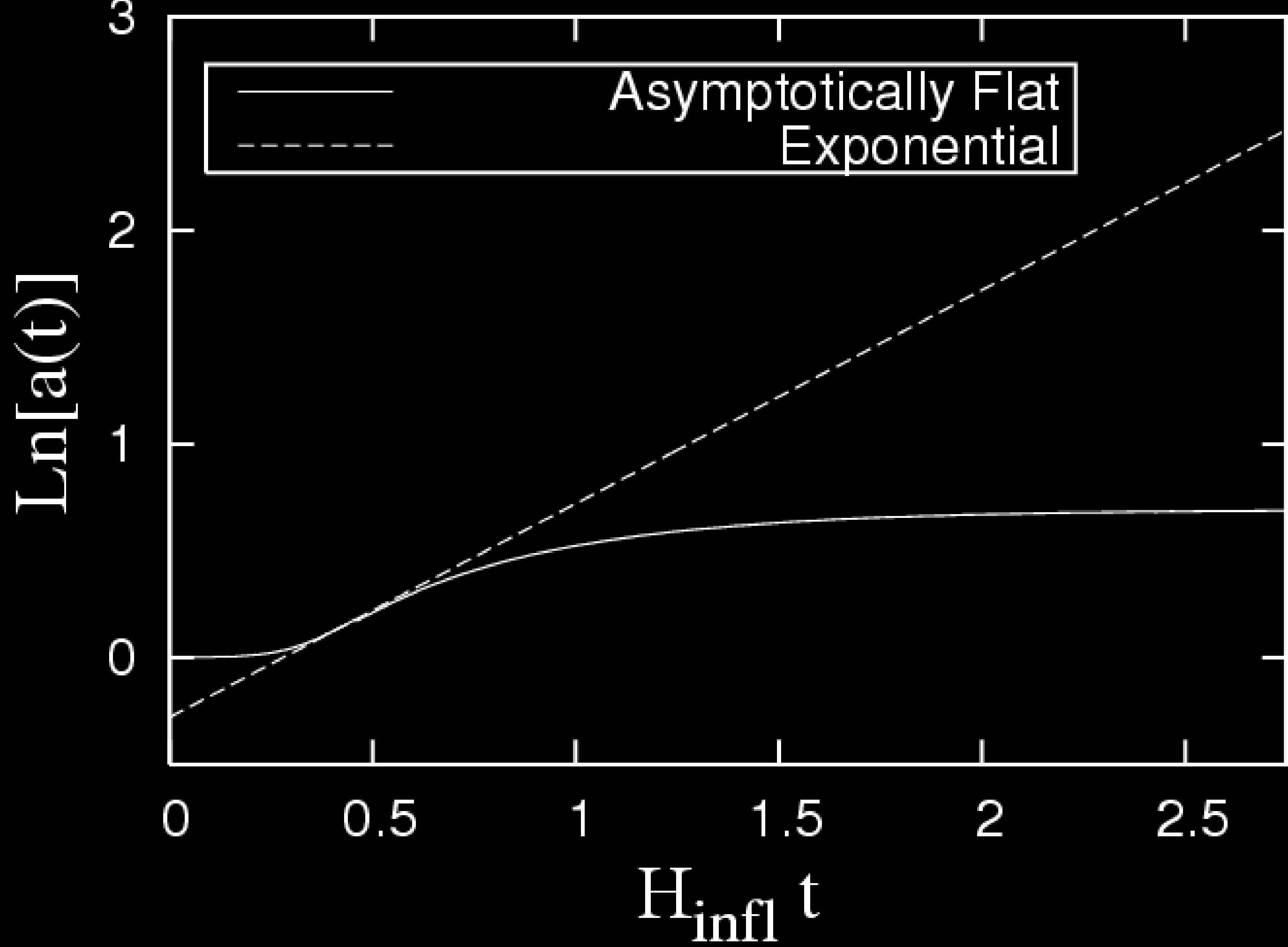
$$d\tau \equiv a(t)^{-3} dt$$

$$a(t(\tau)) = \left[{a_1}^4 + e^{\tau/s}({a_2}^4 - {a_1}^4)(e^{\tau/s}+1)^{-1}\right]^{\frac{1}{4}}$$

P.J. Epstein, *Proc. Nat. Acad. Sciences (US)* **16**, 627 (1930).
C. Eckart, Phys. Rev. **35**, 1303 (1930).
L. Parker, Nature **261**, 20 (1976).



a
sH
2
1
0
-10
-5
0
5
10
$a(\tau)$, $10\,s\,H(\tau)$
τ/s
← Not present in
a(t) ~ c t^x
S=1


3
2
1
0
Ln[a(t)]
Asymptotically Flat
Exponential
0
0.5
1
1.5
2
2.5
H_{infl} t

C^2 Matching Conditions:

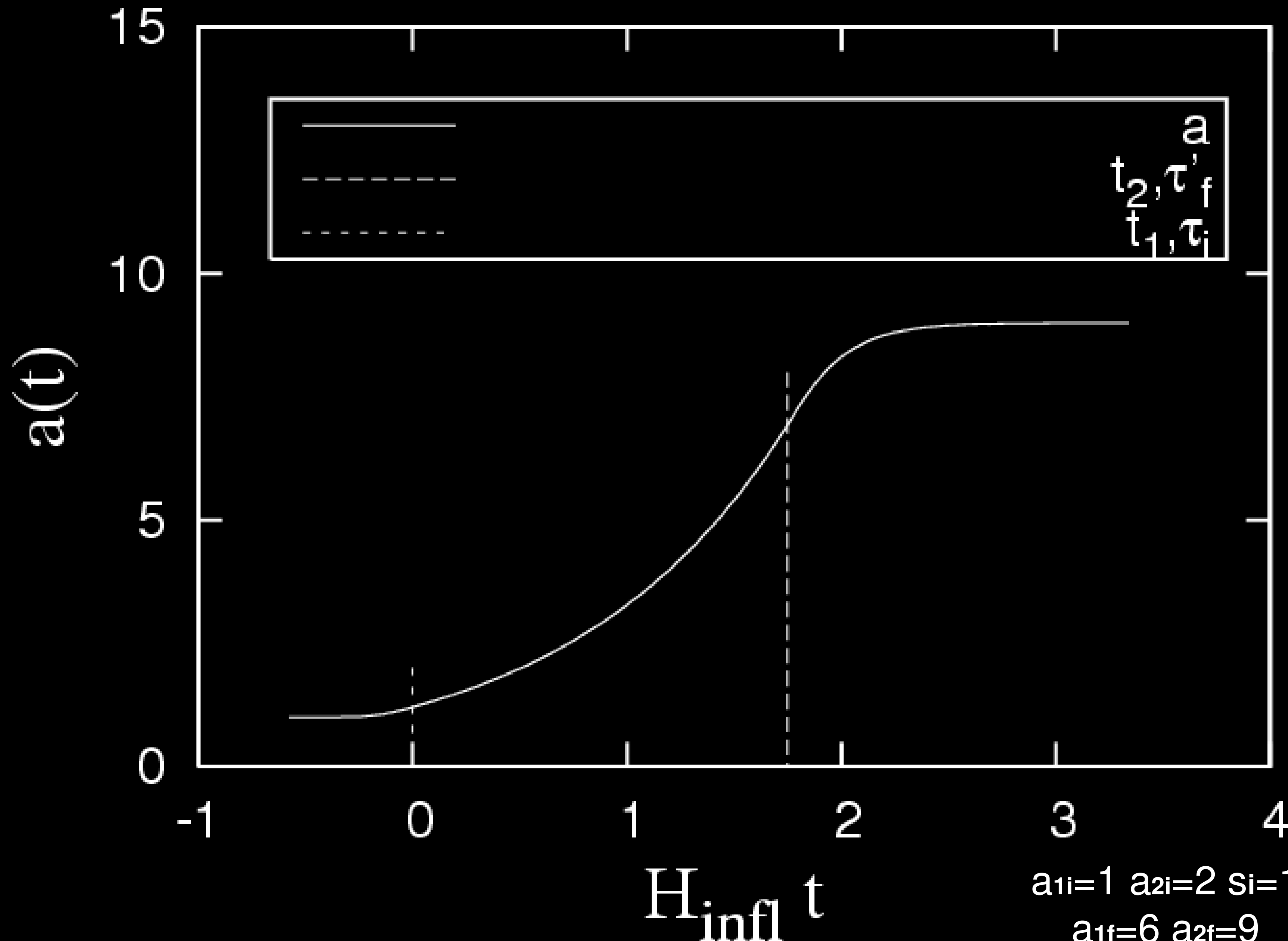
Leads to divergence-free energy density.

(No ultraviolet cutoff frequency necessary.)

$$\tau_{\text{join}} = s \ln\left(\frac{3{a_1}^4 - 3{a_2}^4 + \sqrt{9a_18 + 46{a_1}^4{a_2}^4 + 9a_2^8}}{8{a_2}^4}\right)$$

$$a(\tau_{\text{join}}) = \left(\frac{-3{a_1}^4 - 3{a_2}^4 + \sqrt{9{a_1}^8 + 46{a_1}^4{a_2}^4 + 9{a_2}^8}}{2}\right)^{1/4}$$

$$\begin{aligned} H_{\text{infl}} &= \left[\frac{2^{3/4}\left(-{a_1}^4 + {a_2}^4\right)}{{a_2}^4\left(11{a_1}^4 - 3{a_2}^4 + \sqrt{9{a_1}^8 + 46{a_1}^4{a_2}^4 + 9{a_2}^8}\right)^2 s}\right] \\ &\quad \times \left(-3{a_1}^4 - 3{a_2}^4 + \sqrt{9{a_1}^8 + 46{a_1}^4{a_2}^4 + 9{a_2}^8}\right)^{1/4} \\ &\quad \times \left(3{a_1}^4 - 3{a_2}^4 + \sqrt{9{a_1}^8 + 46{a_1}^4{a_2}^4 + 9{a_2}^8}\right) \end{aligned}$$



a_{1i}=1 a_{2i}=2 s_i=1
a_{1f}=6 a_{2f}=9

EARLY- AND LATE-TIME VACUUMS ARE RELATED BY A BUGOLIUBOV TRANSFORMATION.

$$a_{\vec{k}} = \alpha_k A_{\vec{k}} + \beta_k^* A_{-\vec{k}}^\dagger$$

$$\left\langle N_{\vec{k}} \right\rangle_{t\to\infty} = \langle 0 | \, a_{\vec{k}}^\dagger a_{\vec{k}} \, | 0 \rangle = \left| \beta_k \right|^2$$

L. Parker, *The creation of particles by the expanding universe,*
Ph.D. thesis, Harvard University (1966).

Perturbations to inflaton field are taken to be quantum fluctuations.

$$\phi(\vec{x},t) = \phi(t) + \delta\phi(\vec{x},t)$$

$$\delta\phi = V^{-\frac{1}{2}} \sum_{\vec{k}} \left[A_{\vec{k}} \psi_k(t) e^{i\vec{k}\cdot\vec{x}} + \mathrm{H.c.} \right]$$

Massless Boundary Conditions:

$$\lim_{\tau \to -\infty} \psi_k(\tau) \sim \frac{1}{\sqrt{2ka_{\mathrm{init}}^2}} e^{-ika_{\mathrm{init}}^2 \tau}$$

$$\lim_{\tau \to \infty} \psi_k(\tau) \sim \frac{1}{\sqrt{2ka_{\mathrm{end}}^2}} \left[\alpha_k e^{-ika_{\mathrm{end}}^2 \tau} + \beta_k e^{ika_{\mathrm{end}}^2 \tau} \right]$$

EVOLUTION EQUATION

(Solutions matched in $\psi_k(t)$ and $\psi_k(t)'$ at joining between scale factor segments.)

$$\partial_t{}^2\delta\phi + 3H\partial_t\delta\phi - a^{-2}(t)\sum_{i=1}^{3}\partial_i{}^2\delta\phi + m(\phi^{(0)})^2\delta\phi = 0$$

$$m(\phi^{(0)})^2 = \frac{d^2V}{d(\phi^{(0)})^2}$$

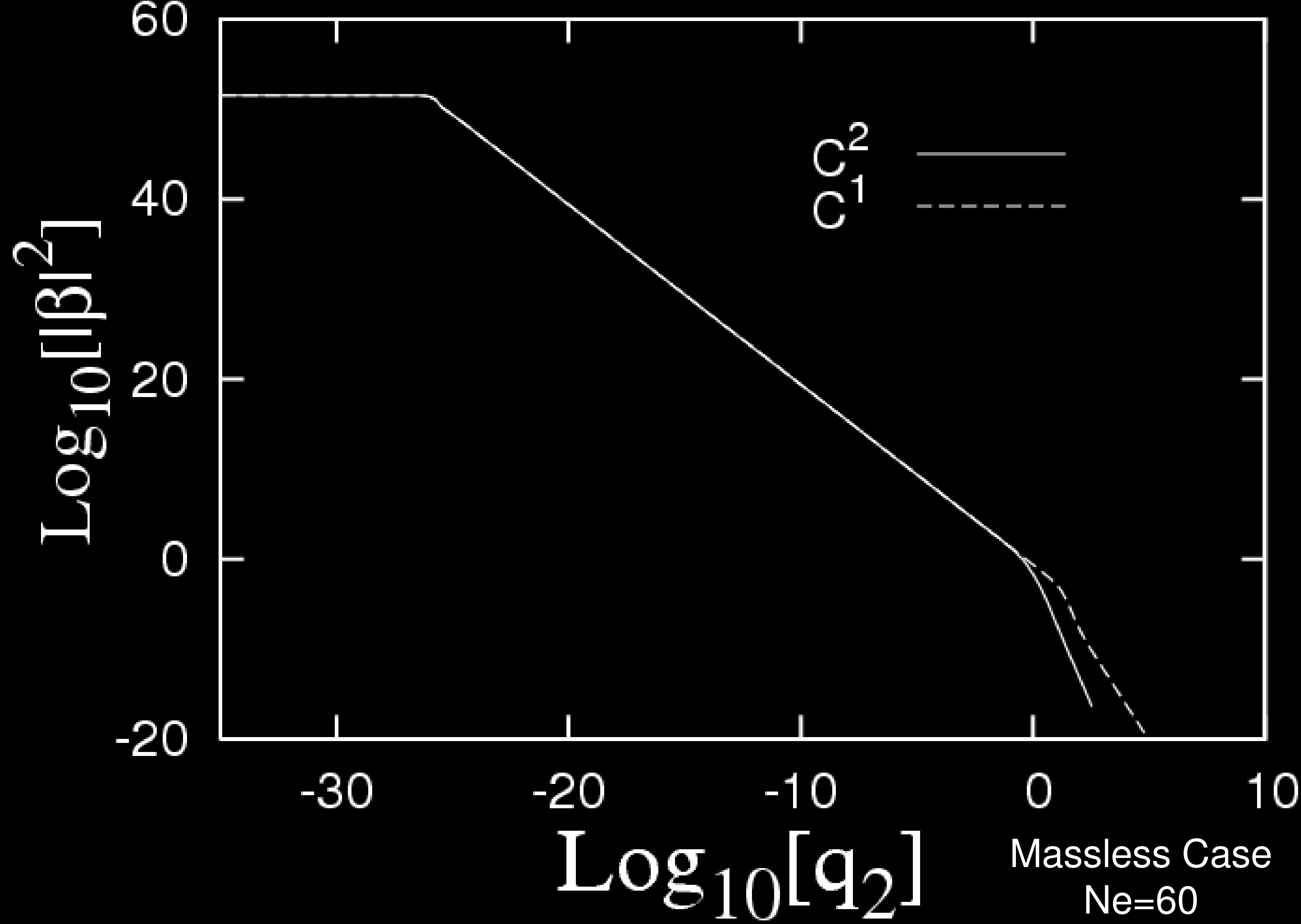
Solutions to Evolution Equation:

Hypergeometric functions in asymptotically flat segments of the scale factor. (massless case shown)

$$\begin{aligned}\psi_k \;=\;& N_1 e^{-ika_1{}^2\tau} F(-ika_1{}^2 s + ika_2{}^2 s,\\ & -ika_1{}^2 s - ika_2{}^2 s; 1 - 2ika_1{}^2 s; -e^{\frac{\tau}{s}})\\ & + N_2 e^{ika_1{}^2\tau} F(ika_1{}^2 s + ika_2{}^2 s,\\ & ika_1{}^2 s - ika_2{}^2 s; 1 + 2ika_1{}^2 s; -e^{\frac{\tau}{s}})\end{aligned}$$

Hankel functions in exponentially growing middle segment of the scale factor. (general case shown)

$$\begin{aligned}\psi_k(t) =\; & a(t)^{-\frac{3}{2}} \left[E(k) H^{(1)}_{\sqrt{\frac{9}{4} - m_H{}^2}} \left(\frac{k}{a(t) H_{\text{infl}}} \right) \right.\\ & \left. + F(k) H^{(2)}_{\sqrt{\frac{9}{4} - m_H{}^2}} \left(\frac{k}{a(t) H_{\text{infl}}} \right) \right]\end{aligned}$$



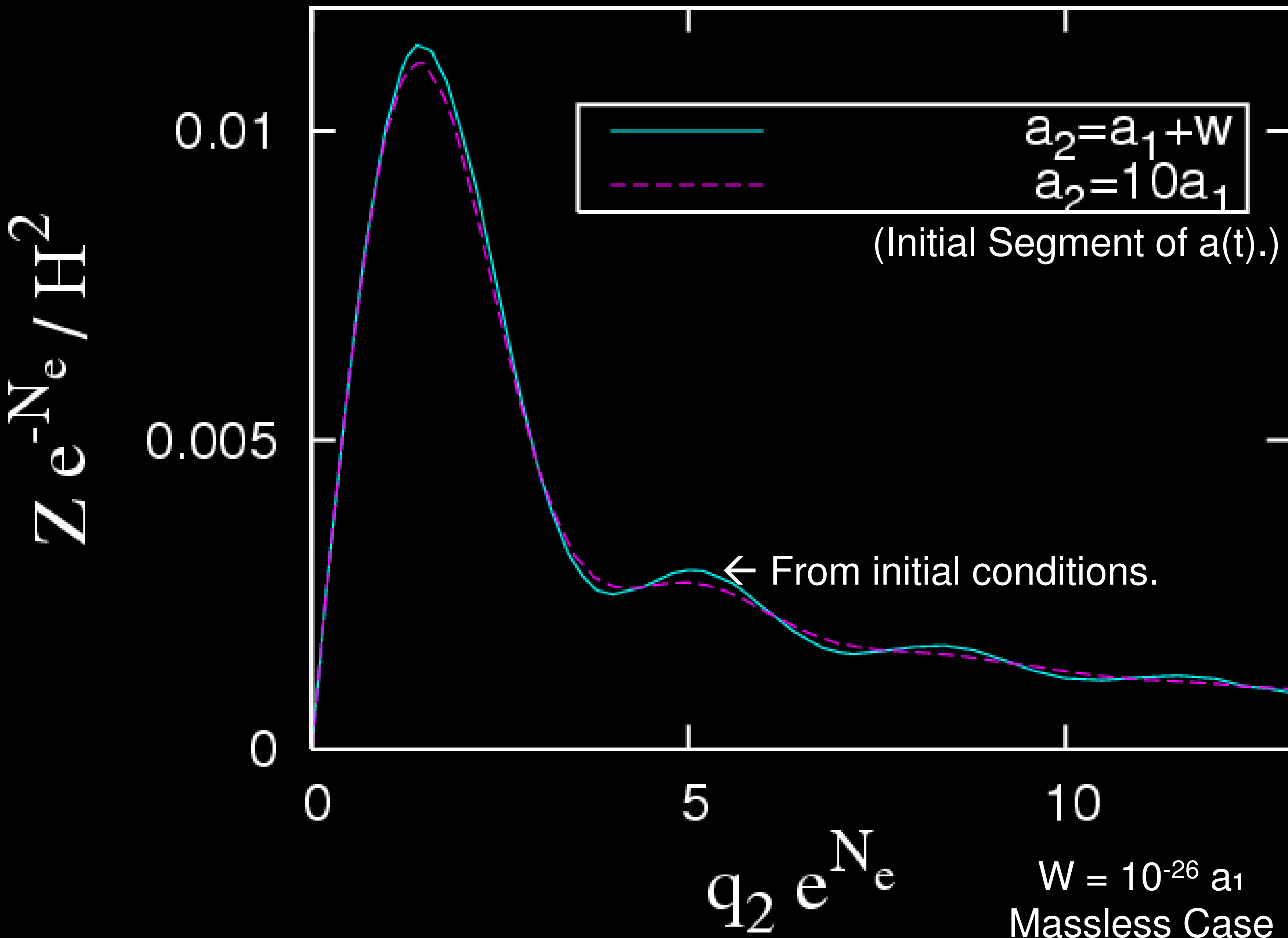
Massless Case
Ne=60

$$\langle\, |\, \delta\phi^2\, |\, \rangle = \frac{1}{2(a_{2f}L)^3} \sum_k \left[\frac{1 + 2|\beta_k|^2}{\sqrt{(k/a_{2f})^2 + m^2}} \right]$$

DISPERSION

$$Z \equiv \frac{q_2 |\beta_{q_2}|^2 H_{infl}^2}{2\pi^2 \sqrt{1 + \frac{m_H^2}{q_2^2}}}$$

$$\langle |\delta\phi^2| \rangle = \int Z \, dq_2$$



$W = 10^{-26}\ a_1$

Massless Case

Two Massive Approximations:

Effective$-k$ Approach

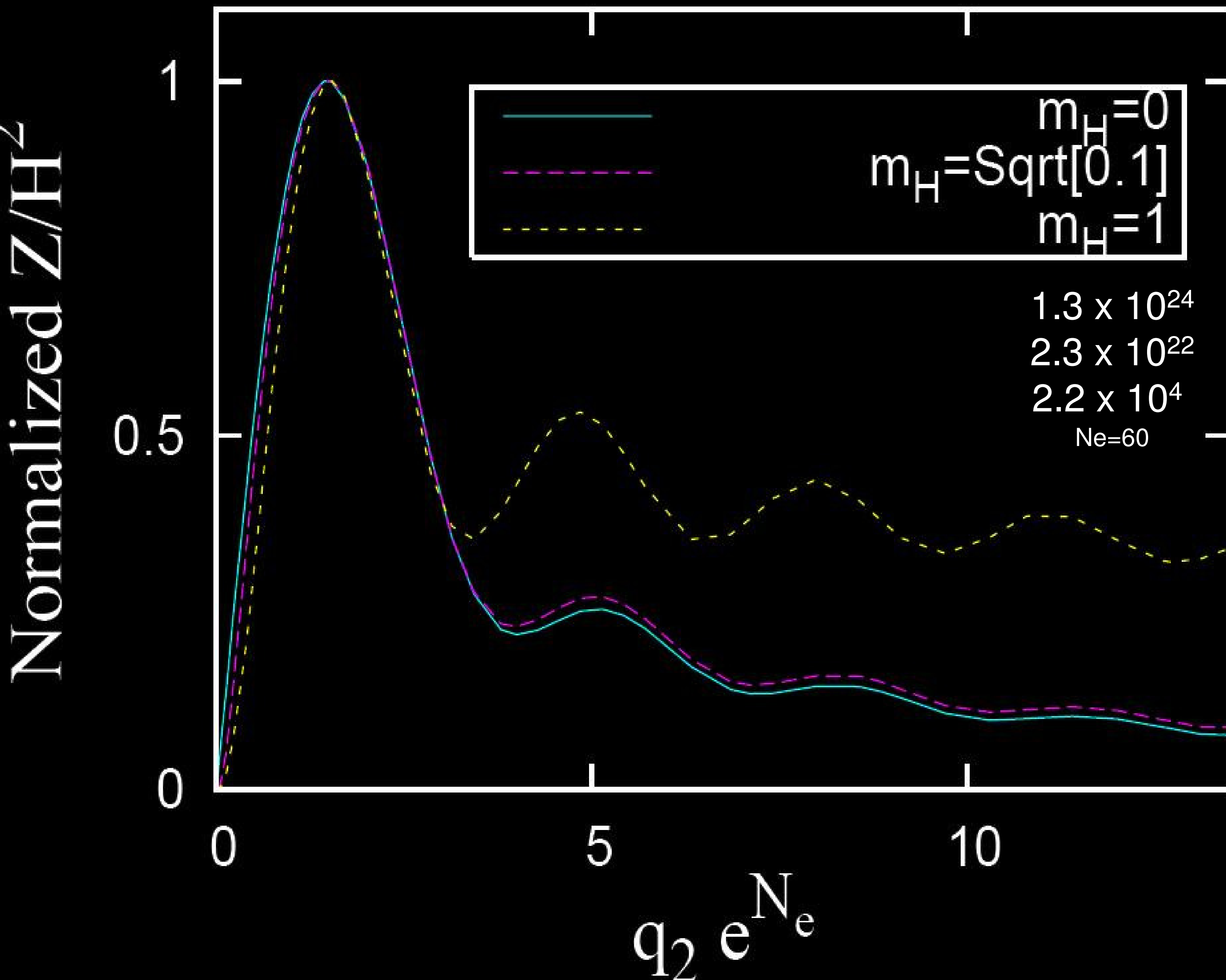
$$k_{\text{effective}} = \begin{cases} \text{Initial}: \sqrt{k^2 + m^2 a_{\text{init}}^2} \\ \text{Final}: \sqrt{k^2 + m^2 a_{\text{end}}^2} \end{cases}$$

Dominant$-$Term Approach

$$k \rightarrow m$$

$${a_1}^2 \rightarrow {a_1}^3$$

$${a_2}^2 \rightarrow {a_2}^3$$


Normalized Z/H^2
1
0.5
0
0
5
10
$q_2\, e^{N_e}$
m_H=0
m_H=Sqrt[0.1]
m_H=1
1.3×10^{24}
2.3×10^{22}
2.2×10^{4}
Ne=60

$$m_H = 0: \qquad P = 2$$

$$0.01 \lesssim m_H \lesssim 1.49: \qquad P = \sqrt{9 - 4{m_H}^2}$$

$$q_2 \lesssim e^{-N_e}: \qquad |\beta_{q_2}|^2 \simeq \frac{1}{4} e^{P N_e}$$

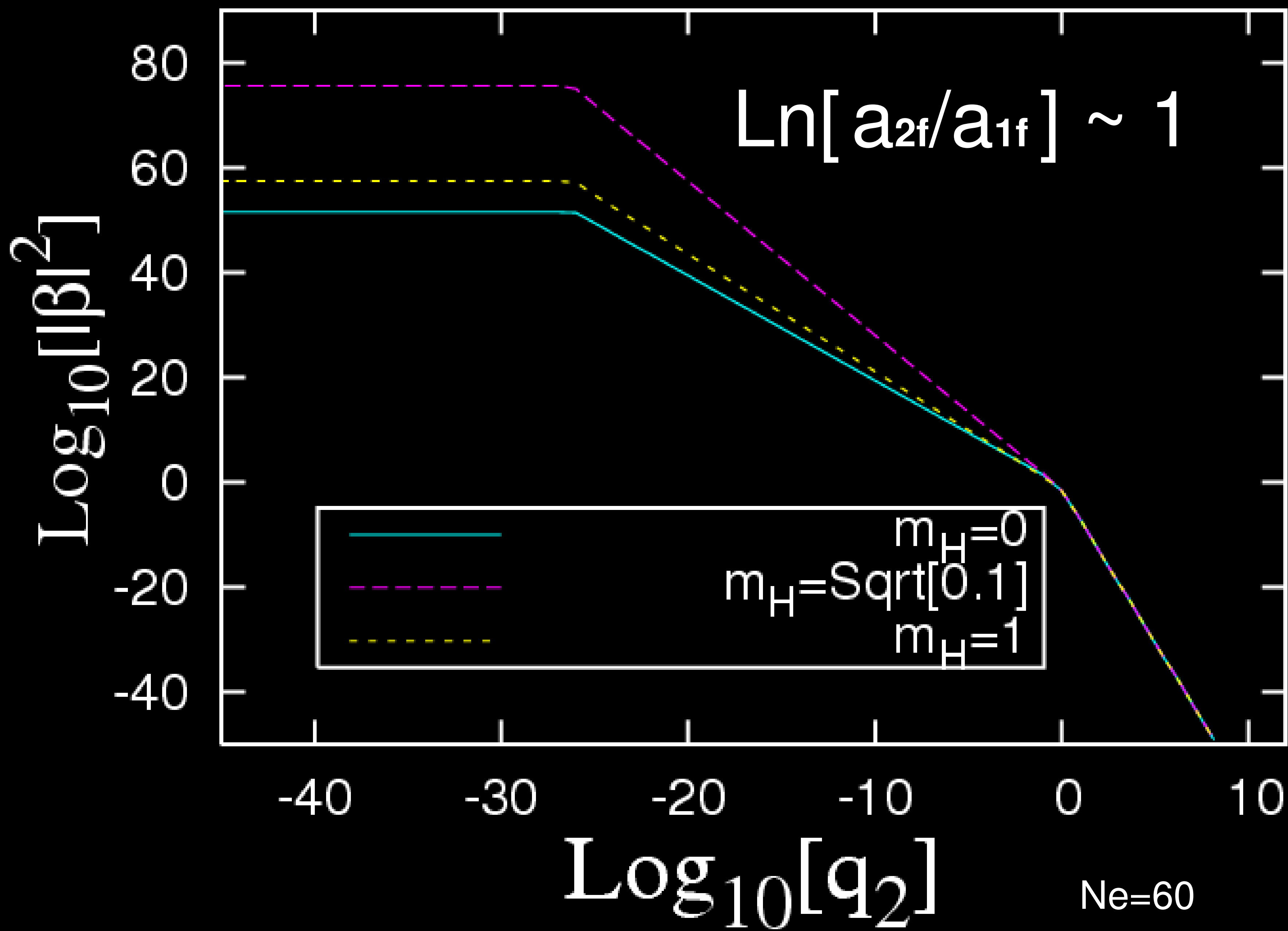
$$e^{-N_e} \lesssim q_2 \lesssim 1^*: \qquad |\beta_{q_2}|^2 \simeq \frac{1}{4} {q_2}^{-P}$$

$$1^* \lesssim q_2: \qquad |\beta_{q_2}|^2 \propto \frac{1}{4} {q_2}^{-6^\dagger}$$

* : 1 Gradual end to inflation.

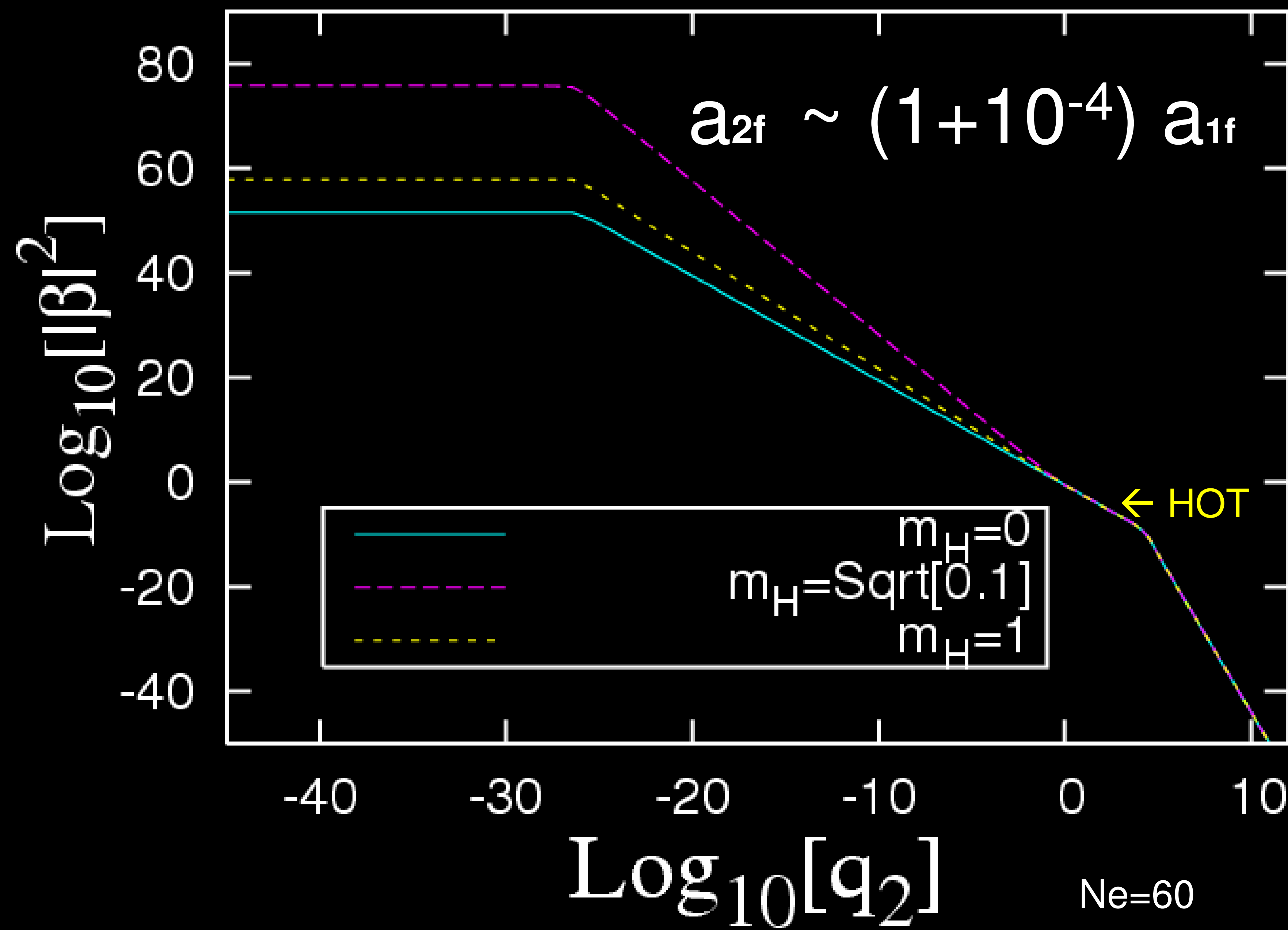
† : -6 $a(t)$ is C^2.

Discontinuities in the scale factor affect the large-mode behavior of particle production.
For a parallel analysis of a harmonic oscillator, see
R.M. Kulsrud, Phys. Rev. **106**, 205 (1957).



$Ln[a_{2f}/a_{1f}] \sim 1$
80
60
40
20
0
-20
-40
$Log_{10}[|\beta|^2]$
-40
-30
-20
-10
0
10
$Log_{10}[q_2]$
$m_H=0$
$m_H=Sqrt[0.1]$
$m_H=1$

Ne=60



$a_{2f} \sim (1+10^{-4})\ a_{1f}$
$Log_{10}[|\beta|^2]$
$Log_{10}[q_2]$
80
60
40
20
0
-20
-40
-40
-30
-20
-10
0
10
$m_H=0$
$m_H=Sqrt[0.1]$
$m_H=1$
← HOT
Ne=60

Reheating from abrupt end to inflation:

depends on parameters of final asymptotically flat segment of the scale factor with C^2 matching.

$$\text{Energy Density} \simeq \frac{H_{\text{inflation}}^4 \left(\frac{a_{2f}}{a_{2f}-a_{1f}}\right)^2}{16\pi^2(c^3/\hbar)}$$

(With gradual transition, temperature of order $T=H/2\pi$.)

Thank You

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