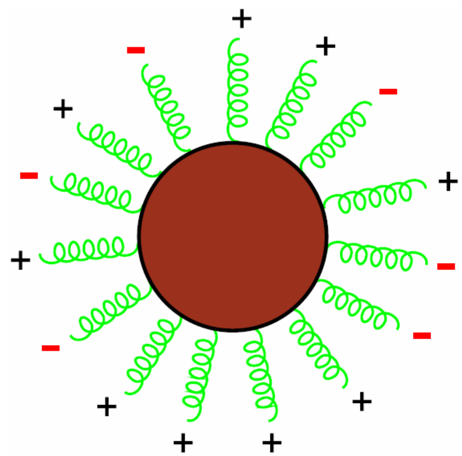
Recent Developments in Perturbative QCD



Lance Dixon, SLAC

DIS 2005

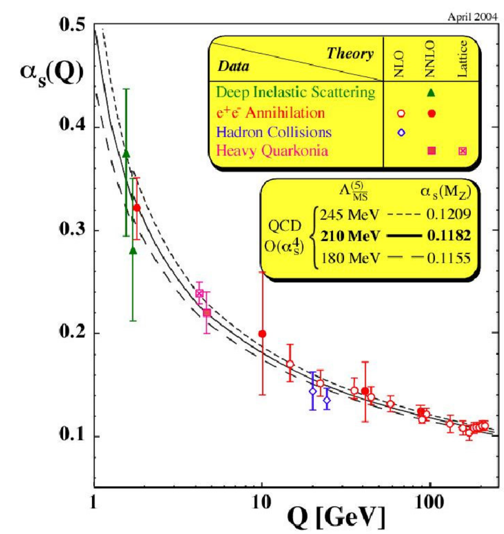
Madison, April 27, 2005

Not a recent development, but …

2004 Nobel Prize in Physics

D. Gross, H. D. Politzer, F. Wilczek

"for the discovery of asymptotic freedom in the theory of the strong interaction"



… without it there would be no perturbative QCD

Two Topics

- Progress in NNLO computations
- Twistor spinoffs to perturbative QCD

A method is more important than a discovery, since the right method will lead to new and even more important discoveries.

- Landau

NNLO Progress

NNLO QCD predictions required for precise extraction from collider data of:

- α_s
- parton distributions
- electroweak parameters
- LHC (partonic) luminosity
- Higgs couplings

- NNLO progress in terms of number of scales: 0,1,2, ∞
- More scales tougher, but more flexible applications

No-scale, inclusive problems

$R(e^+e^- \to \text{hadrons})$ & $R(\tau \to \nu_\tau + \text{hadrons})$

Gorishnii, Kataev, Larin; Surguladze, Samuel (1990)

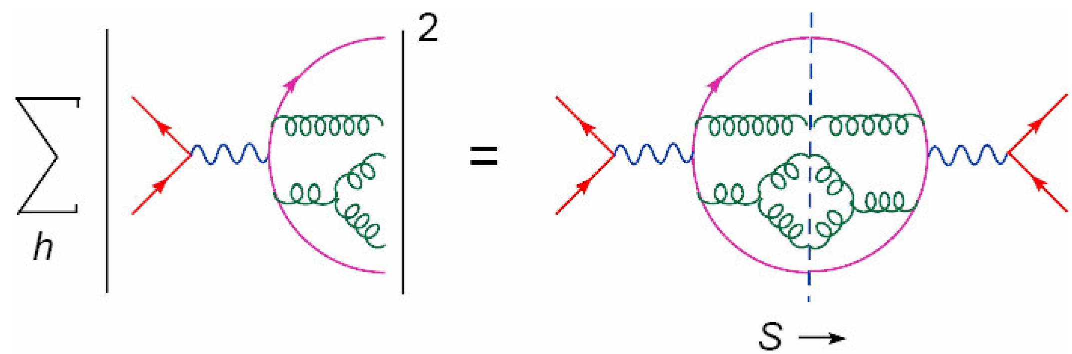
DIS sum rules: $\int_0^1 dx\, F_n(x)$

Bjorken $(\bar{\nu}p - \nu p)$ Larin, Tkachov, Vermaseren (1990)

Bjorken $(\vec{e}p)$ & Gross-Llewellyn-Smith $(\nu p + \bar{\nu}p)$

Larin, Vermaseren (1991)

Unitarity $\Rightarrow$
propagator-type loop integrals;
s (or Q^2) factors out



No-scale problems (cont.)

- Multi-loop integral technology: Integration by parts (IBP)

Chetyrkin, Tkachov (1981)

$$0 = \int d^D p\, d^D q \ldots \frac{\partial}{\partial q^\mu} \frac{k^\mu}{p^2 q^2 (p+q)^2 \ldots}$$

- Reduces problem to system of linear equations, solved recursively by MINCER, in terms of few “master integrals”

Gorishnii, Larin, Surguladze, Tkachov (1989)

No-scale $\Rightarrow$ analytic simplicity -- pure numbers

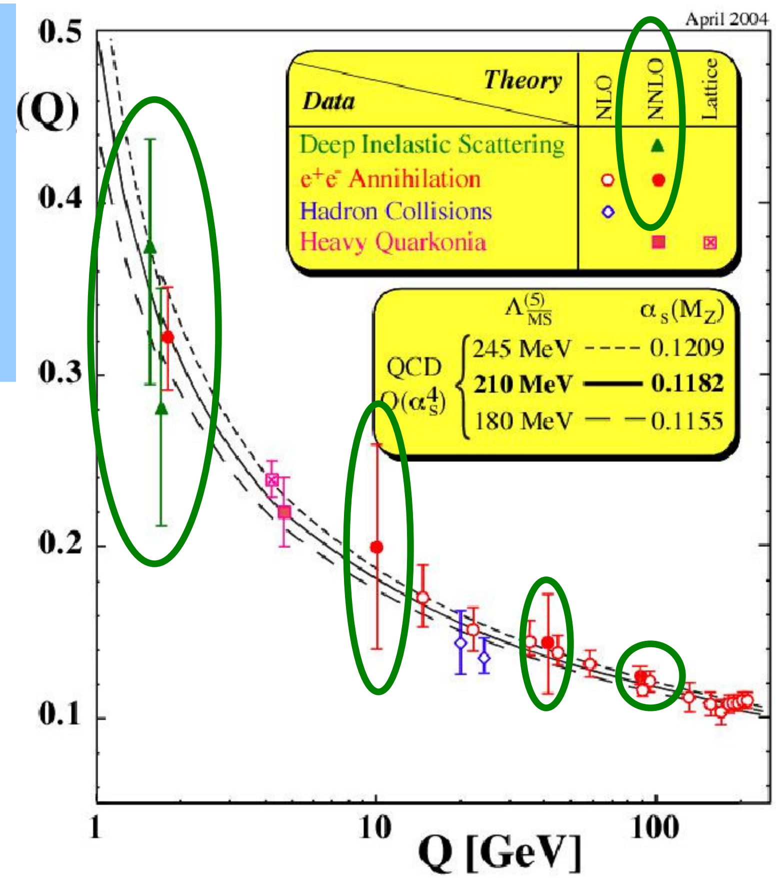
$$\begin{aligned}\frac{R_{e^+e^-}}{R^{(0)}} = {} & 1 + \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 \left[-11\zeta(3) + \frac{365}{24} + n_f\left(\frac{2}{3}\zeta(3) - \frac{11}{12}\right)\right] \\ & + \left(\frac{\alpha_s}{\pi}\right)^3 \left[\frac{275}{6}\zeta(5) - \frac{1103}{4}\zeta(3) - \frac{121}{8}\zeta(2) + \frac{87029}{288}\right. \\ & + n_f\left(-\frac{25}{9}\zeta(5) + \frac{262}{9}\zeta(3) + \frac{11}{6}\zeta(2) - \frac{7847}{216}\right) \\ & \left. + n_f^2\left(-\frac{19}{27}\zeta(3) - \frac{1}{18}\zeta(2) + \frac{151}{162}\right)\right]\end{aligned}$$

No-scale applications

- Have led to α_s determinations with least theoretical uncertainty (if Q^2 large enough)

- Experimental precision stressed by leading "1":

$$R \propto 1 + \frac{\alpha_s}{\pi} + \cdots$$



1-scale, inclusive problems

Drell-Yan, W, Z total cross section

$\sigma^{\text{tot}}(pp \to V + X)$ Hamberg, van Neerven, Matsuura (1990)

Higgs total cross section $(m_t \to \infty)$

$\sigma^{\text{tot}}(pp \to H + X)$ Harlander, Kilgore; Anastasiou, Melnikov (2002); Ravindran, Smith, van Neerven (2003)

$\hat{\sigma}$ depends on $z = M_{V,H}^2/\hat{s}$

DIS coefficient functions $C_i(z)$ Van Neerven, Zijlstra (1991)

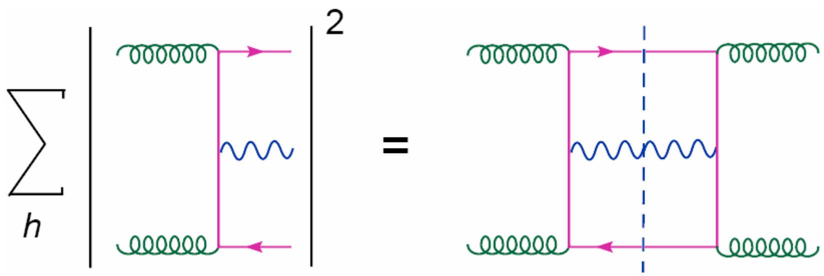
F_L -- Moch, Vermaseren, Vogt (2004)

Leading-twist anomalous dimensions

DGLAP kernels $P_{ij}(x)$ Moch, Vermaseren, Vogt (2004)

1-scale problems (cont.)

- Can apply unitarity and multi-loop integral technology to DY/Higgs production too: forward scattering instead of propagator Anastasiou, Melnikov (2002)



DIS very similar.
Fixed moment $N \Rightarrow$ MINCER
Variable $N \Rightarrow$ new algorithms

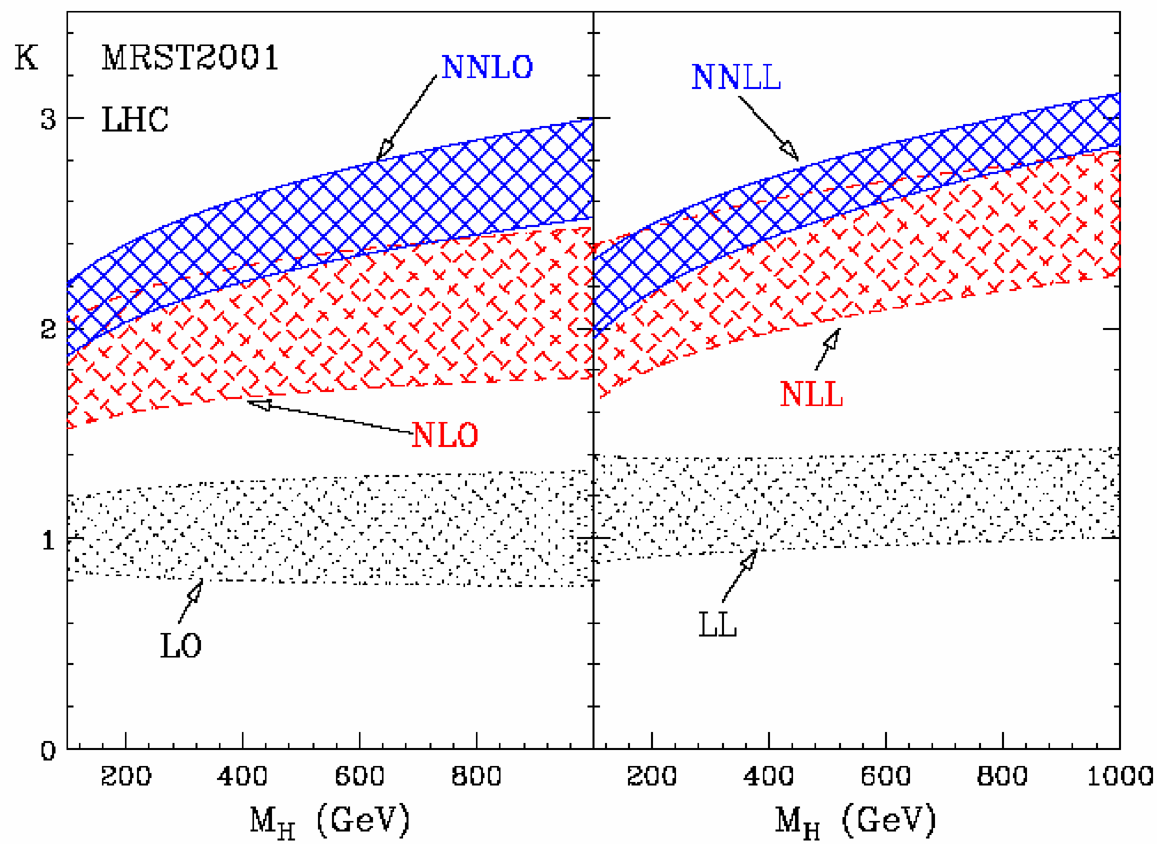
- 1-scale analytic structure: (harmonic) polylogarithms $\mathrm{Li}_n(z)$

- Or, use Mellin transform $\int_0^1 dz\, z^{N-1} f(z)$ to convert to harmonic sums $S_k(N) = \sum_{i=1}^{N} \frac{1}{i^k}$ commonly used in DIS

Blumlein, Ravindran (2005)

1-scale applications

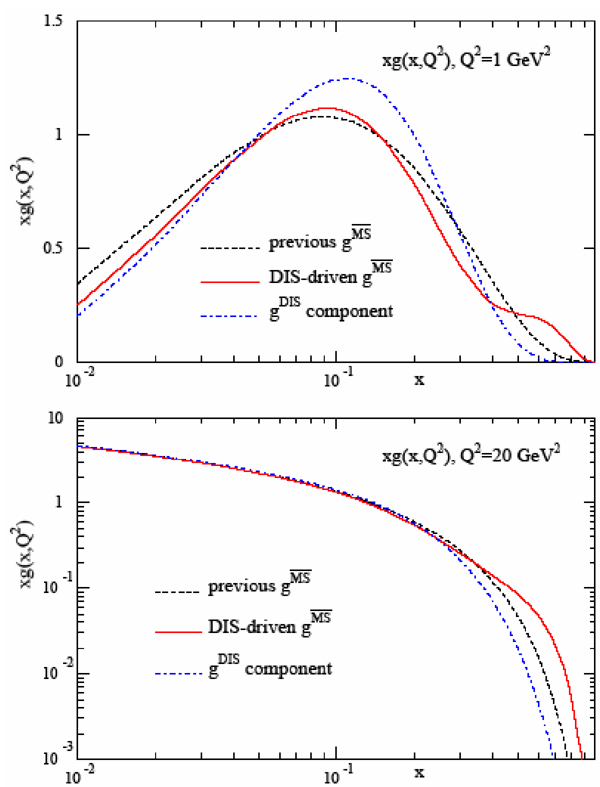
- Reduction of uncertainty on Higgs total cross section at LHC

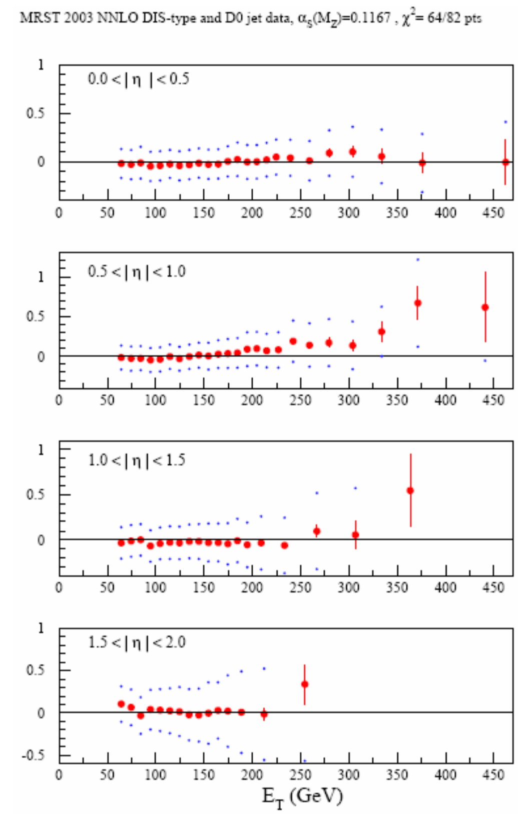


Harlander, Kilgore;
Anastasiou, Melnikov;
Ravindran, Smith,
van Neerven;
Catani, De Florian,
Grazzini, Nason;
Grazzini, hep-ph/0209032

1-scale applications (cont.)

- NNLO fits to pdfs, for example: "DIS-driven" MRST vs. D0 jet data:





Martin, Roberts, Stirling, Thorne, hep-ph/0410230

Caveat: jet partonic cross section still at NLO

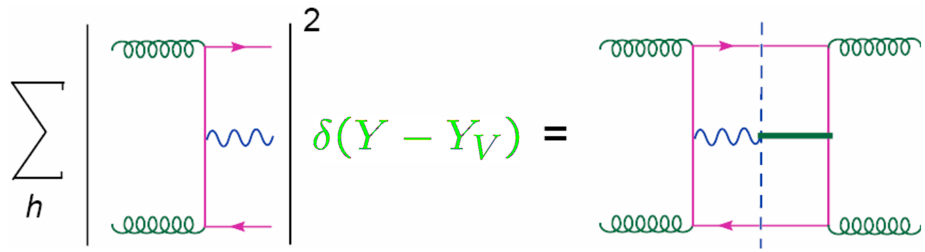
2-scale, semi-inclusive problem

Drell-Yan, W, Z rapidity distribution

$\frac{d\sigma(pp\to V+X)}{dY_V}$ Anastasiou, LD, Melnikov, Petriello (2003)

$\hat{\sigma}$ depends on $z = M^2_{V,H}/\hat{s}$ and Y_V

- Unitarity + multi-loop IBP still works, with one more "propagator" to implement rapidity δ function

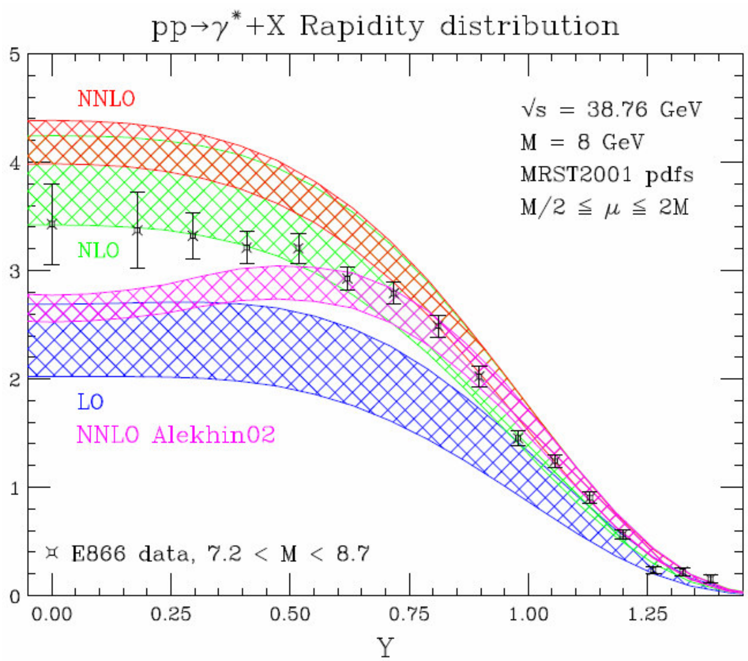


- Complicated analytic structure: $\mathrm{Li}_2\left(\frac{u-1-i\sqrt{(4u^2-z(1+u)^2)/z}}{2u}\right)$

$u \equiv \frac{x_1}{x_2}\exp(-2Y_V)$

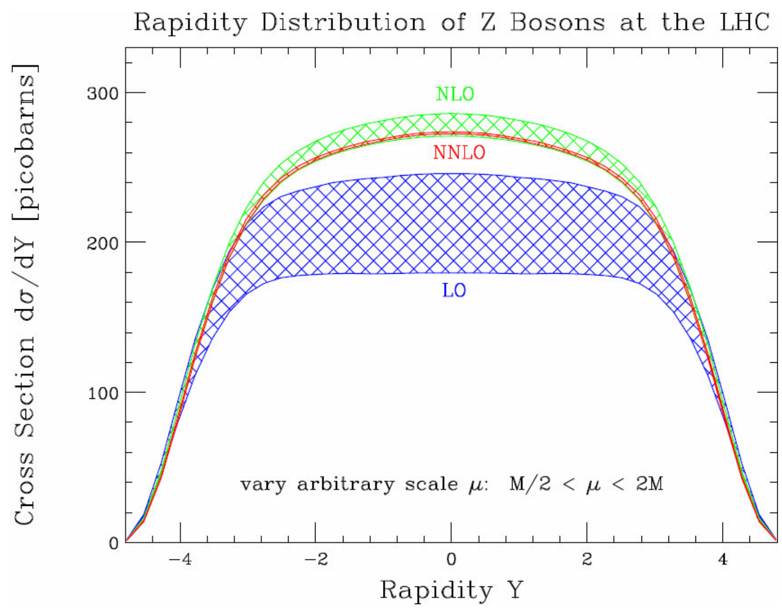
2-scale applications

- Fixed target DY for $\overline{q}(x)$



$$\text{At LO, } x_1 = \frac{M}{\sqrt{s}} \exp(Y),\ x_2 = \frac{M}{\sqrt{s}} \exp(-Y)$$

- Collider W/Z for "partonic luminosity monitor" Dittmar, Pauss, Zurcher (1997)



Anastasiou, LD, Melnikov, Petriello (2003)

∞-scale problem

- "Holy grail": Flexible method for arbitrary (infrared-safe) observable at NNLO.

Partial wish list:

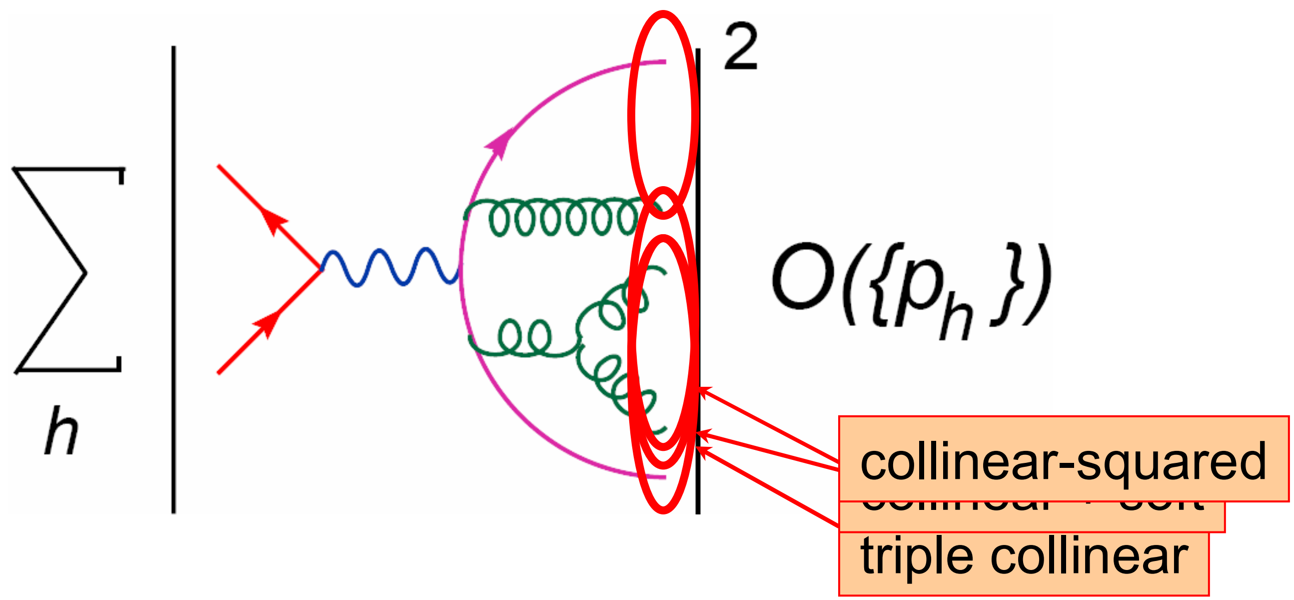
- e^+e^- event-shape observables
- pp or ep $\to$ inclusive jets, dijets, multijets
- $pp \to (W, Z, H) + X$ with parton-level cuts
- $pp \to (W, Z)$ + jets

- Analytic structure too complicated; go numerical

Numerical phase-space integration

- Integration has to be done in $D=4-2\varepsilon$ due to severe infrared divergences ($1/\varepsilon^4$)

Example: e^+e^- event-shapes



Phase-space integration (cont.)

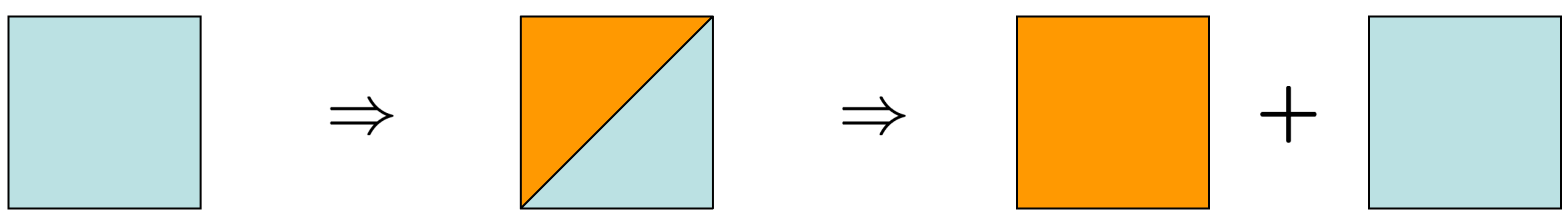
Two basic approaches at present:

Method 1. Iterated sector decomposition

Partition integration region and remap to make divergences "1-dimensional", let computer find subtraction terms

Binoth, Heinrich; Anastasiou, Melnikov, Petriello (2003,2004)

Simple example: $I = \int_0^1 \int_0^1 dx dy \frac{x^\epsilon y^\epsilon}{(x+y)^2}$



Phase-space integration (cont.)

Method 2. Use known factorization properties of amplitudes to build subtraction terms for general processes, and integrate them

Many authors (~1997-2003)

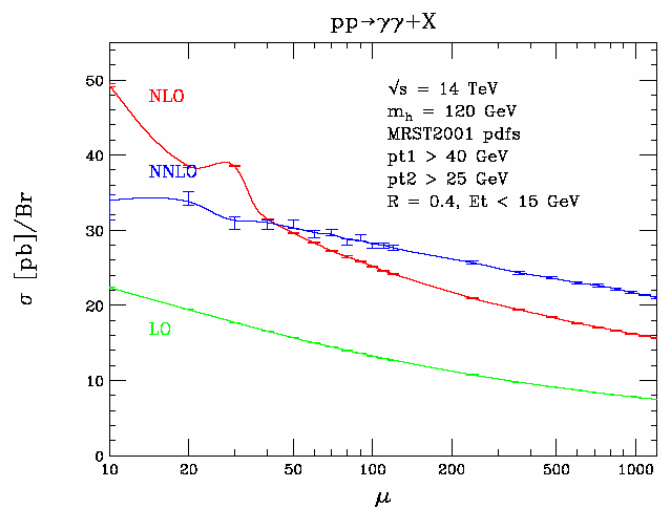
Kosower; Weinzierl; Gehrmann, Gehrmann-de Ridder, Heinrich (2003)
Frixione, Grazzini (2004); Gehrmann, Gehrmann-de Ridder, Glover (2004,2005);
Del Duca, Somogyi, Trocsanyi (2005)

- construct the 2-unresolved-parton counterterm using the IR currents

$$
\begin{aligned}
\mathbf{A}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r \sum_{s\neq r} \Bigg\{ \sum_{i\neq r,s} \Bigg[& \frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s} \frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{S}_{rs} \\
& + \frac{1}{2}\Big(\mathbf{CS}_{ir;s} - \mathbf{C}_{irs}\mathbf{CS}_{ir;s} - \sum_{j\neq i,r,s} \mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} \Big) \Bigg] \\
& - \sum_{i\neq r,s} \Bigg[\mathbf{CS}_{ir;s}\mathbf{S}_{rs} + \mathbf{C}_{irs}\Big(\frac{1}{2}\mathbf{S}_{rs} - \mathbf{CS}_{ir;s}\mathbf{S}_{rs}\Big) \\
& + \sum_{j\neq i,r,s} \mathbf{C}_{ir;js}\Big(\frac{1}{2}\mathbf{S}_{rs} - \mathbf{CS}_{ir;s}\mathbf{S}_{rs}\Big) \Bigg] \Bigg\} |\mathcal{M}^{(0)}_{m+2}|^2
\end{aligned}
$$

∞-scale applications

- $pp \to H + X \to \gamma\gamma + X$ with parton-level cuts



Anastasiou, Melnikov, Petriello (2004,2005)

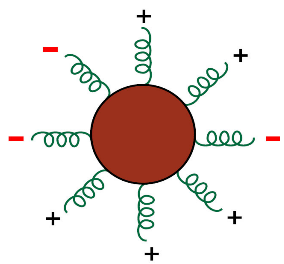
- e^+e^- event-shapes:
  Abelian (C_F^2) part of $\langle 1 - T \rangle$: -20.4 $\pm$ 4
  Gehrmann-De Ridder, Gehrmann, Glover (2004)

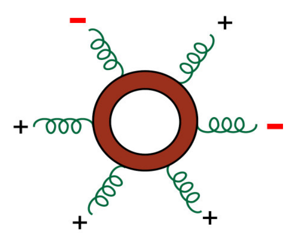
Tip of the iceberg!

And now for something completely different …

Twistor spinoffs to perturbative QCD

Motivation: Make computations of QCD
trees and loops more efficient





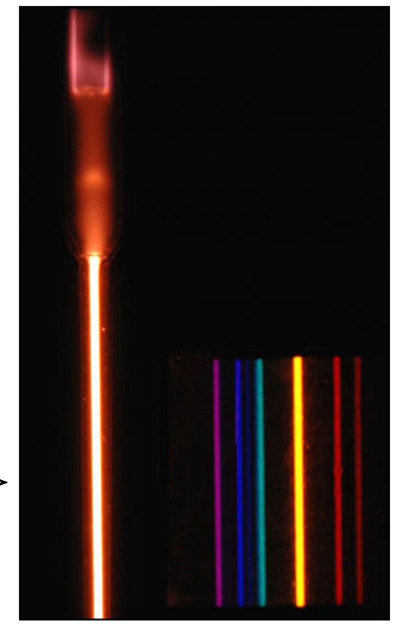
by understanding analytic structure
(sometimes hidden)

Simplicity in Fourier space

Example of atomic spectroscopy

t

$\int dt \, \exp(i\omega t) \quad \Rightarrow$



ω

lines appear

Twistor space

Penrose (1967);
Witten (2003)

Describe momentum space using spinors, not Lorentz vectors k_i^μ

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$ left-handed: $(\tilde{\lambda}_i)_{\dot{\alpha}} = u_-(k_i)$

$$k_i^\mu (\gamma_\mu)_{\alpha\dot{\alpha}} = (\not{k}_i)_{\alpha\dot{\alpha}} = u_+(k_i) u_-(k_i) = (\lambda_i)_\alpha (\tilde{\lambda}_i)_{\dot{\alpha}}$$

Fourier transform $\tilde{\lambda}_i$, but not λ_i, for each leg i

$$\tilde{\lambda}_{\dot{a}} = i \frac{\partial}{\partial \mu^{\dot{a}}} \qquad \mu^{\dot{a}} = -i \frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}}$$

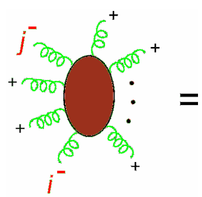
Twistor space homogeneous coordinates: $(\lambda_1, \lambda_2, \mu^{\dot{1}}, \mu^{\dot{2}})$
for each i

Amplitudes $A(k_i) \Rightarrow A(\lambda_i, \tilde{\lambda}_i) \Rightarrow A(\lambda_i, \mu_i)$

Twistor transform in QCD

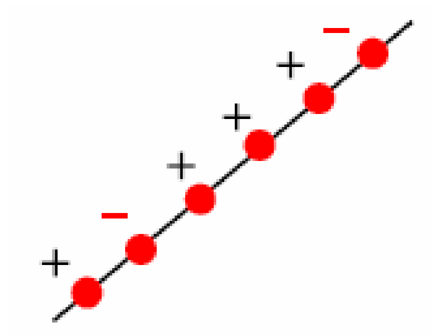
Witten (2003)

Parke-Taylor formula (1986):



$$= \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,\delta(\textstyle\sum_i k_i) \quad = \int d^4x\, A(\lambda_i)\, \exp(ix\lambda_i\tilde{\lambda}_i)$$

$$\int d\tilde{\lambda}_i\, \exp(i\mu_i\tilde{\lambda}_i) \quad \Rightarrow$$



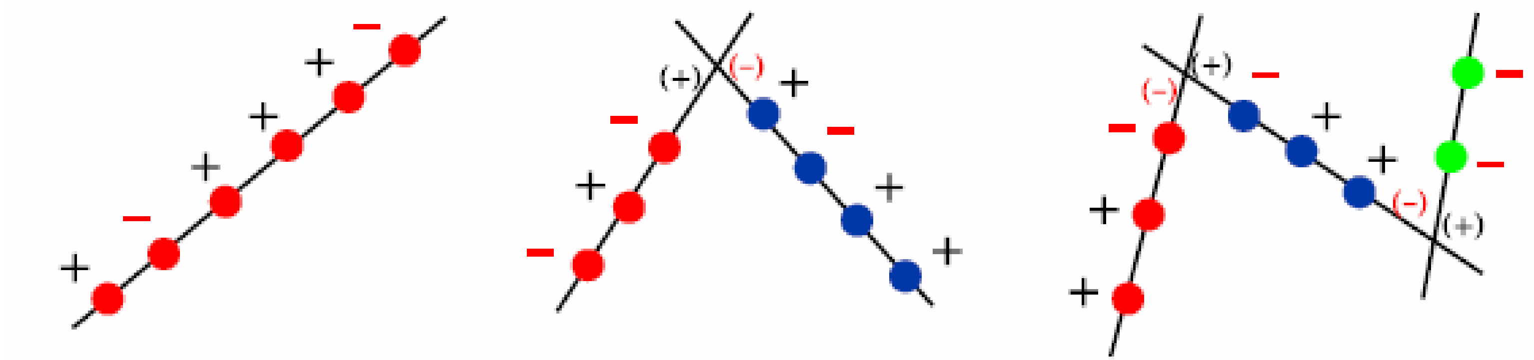
lines appear

Twistor structure of trees

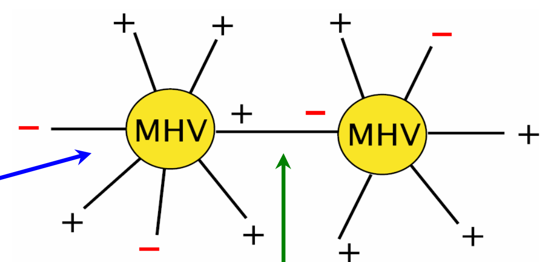
In general:

more lines appear!

$n_- = 2$ (MHV) $n_- = 3$ (NMHV) $n_- = 4$ (NNMHV)



Led to MHV rules:



off-shell MHV (Parke-Taylor) amplitudes

scalar propagator, $1/p^2$

Cachazo, Svrcek, Witten (2004)

MHV rules for trees

Rules are quite efficient, can be extended to:

massless fermions

Georgiou, Khoze, hep-th/0404072;
Wu, Zhu, hep-th/0406146;
Georgiou, Glover, Khoze, hep-th/0407027

Higgs bosons (via *Hgg* effective vertex)

LD, Glover, Khoze, hep-th/0411092;
Badger, Glover, Khoze, hep-th/0412275

vector bosons (W,Z,γ^*),
including DIS multi-jet production

Bern, Forde, Kosower, Mastrolia, hep-th/0412167

Still, each extension requires a little thought;
initial "proofs" of correctness partly based on empirical agreement.

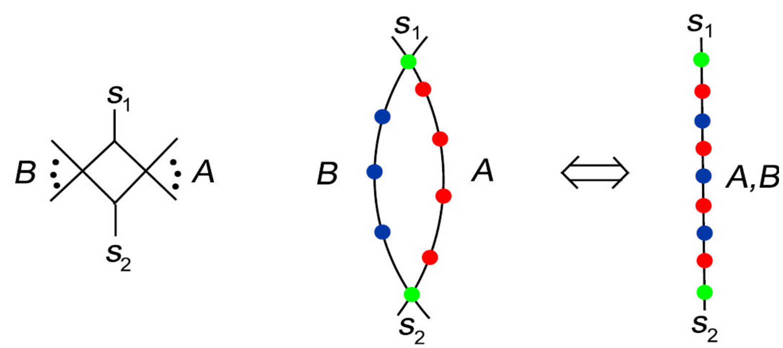
Recent approach to massive quarks a bit different

Schwinn, Weinzierl, hep-th/0503015

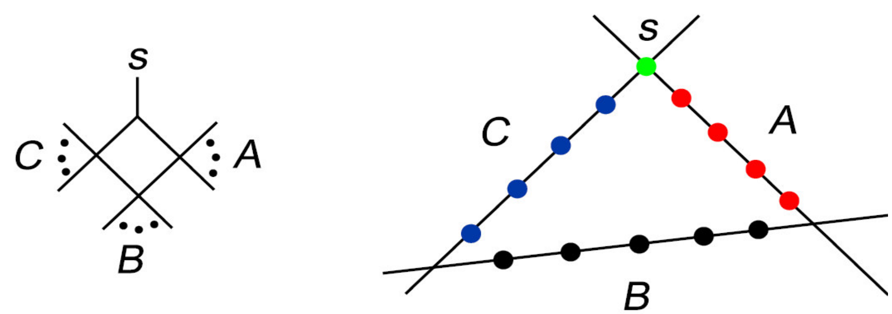
Twistor structure of loops

Simplest for coefficients of box integrals in N=4 super-Yang-Mills:

$n_- = 2$ (MHV)



$n_- = 3$ (NMHV) — nondegenerate

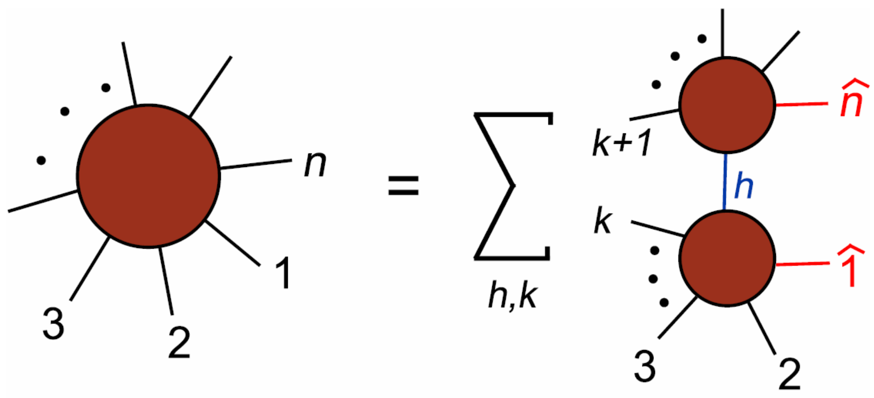


Bern, LD, Kosower (2004)

An on-shell recursion relation for trees

Britto, Cachazo, Feng, hep-th/0412308;
Britto, Cachazo, Feng, Witten hep-th/0501052

$$A_n(1,2,\ldots,n) = \sum_{h=\pm}\sum_{k=2}^{n-2} A_{k+1}(\hat{1},2,\ldots,k,-\hat{K}_{1,k}^{-h}) \times \frac{i}{K_{1,k}^2} A_{n-k+1}(\hat{K}_{1,k}^{h},k+1,\ldots,n-1,\hat{n})$$



A_{k+1} and A_{n-k+1} are on-shell tree amplitudes with fewer legs, evaluated with momenta shifted by a complex amount

Momentum shift

Complex shift best described using spinors, not Lorentz vectors k_i^μ

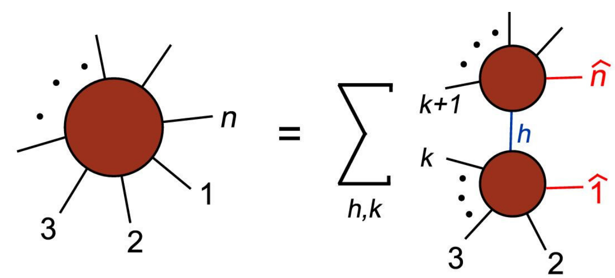
$$k_i^\mu (\gamma_\mu)_{\alpha\dot\alpha} = (\not{k}_i)_{\alpha\dot\alpha} = u_+(k_i) u_-(k_i) = (\lambda_i)_\alpha (\tilde\lambda_i)_{\dot\alpha}$$

Complex null momenta also products of (different) spinors (degenerate 2 x 2 matrix): $0 = p^2 = \det(\not{p}) \quad \Rightarrow \quad (\not{p})_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda'_{\dot\alpha}$

The shift:

$$\hat\lambda_1 = \lambda_1 - \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle} \lambda_n \qquad \hat{\tilde\lambda}_1 = \tilde\lambda_1$$

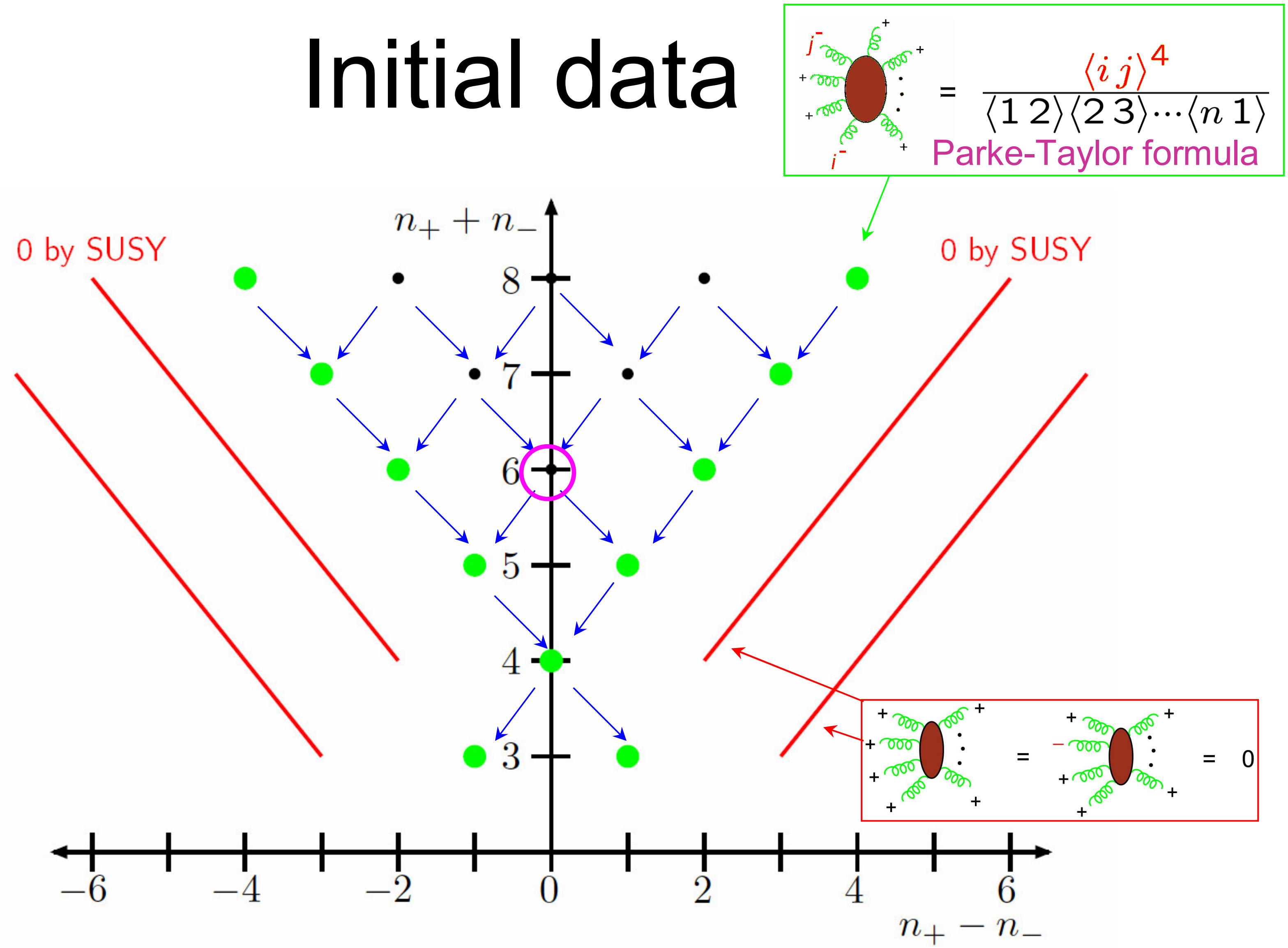
$$\hat\lambda_n = \lambda_n \qquad \hat{\tilde\lambda}_n = \tilde\lambda_n + \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle} \tilde\lambda_1$$



$\not{K}_{1,k}$ also shifted by $\propto \lambda_n \tilde\lambda_1$ to preserve momentum conservation

Proof by Cauchy's thm. -- poles of *A(z)* : $\hat\lambda_1 = \lambda_1 + z\lambda_n \qquad \hat{\tilde\lambda}_n = \tilde\lambda_n - z\tilde\lambda_1$

Initial data



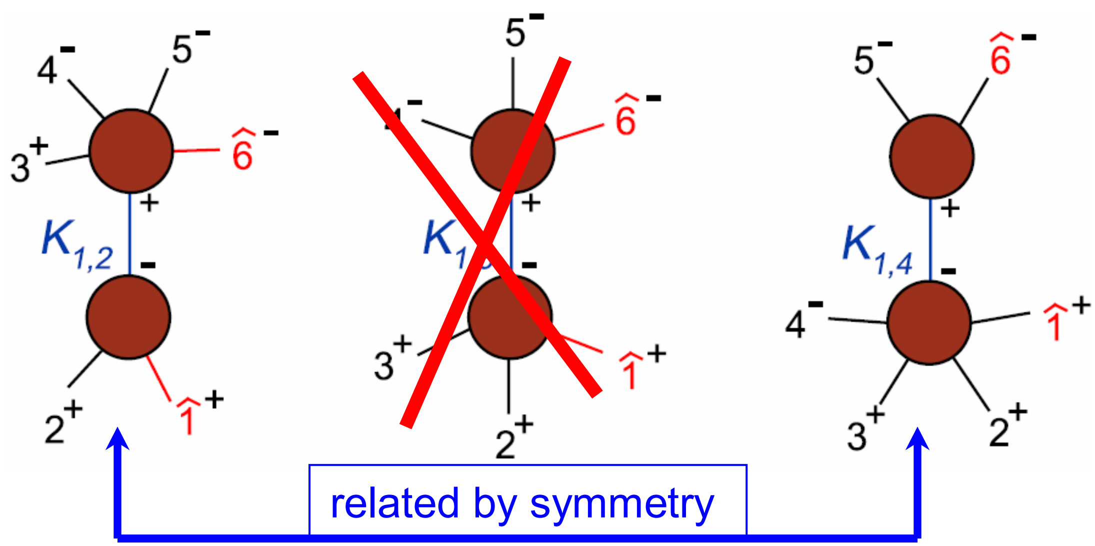
A 6-gluon example

220 Feynman diagrams for *gggggg*

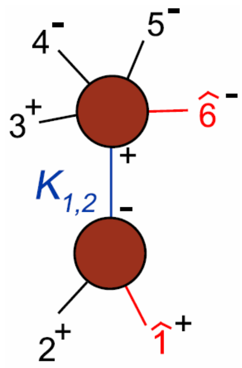
Helicity + color + MHV results + symmetries

$\Rightarrow$ only $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$, $A_6(1^+,2^+,3^-,4^+,5^-,6^-)$

3 BCF diagrams $\rightarrow$ 2 $\rightarrow$ 1



The one $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$ diagram



$$= -\frac{i}{s_{12}} \frac{[\hat{1}\,2]^3}{[2\,\hat{K}][\hat{K}\,\hat{1}]} \frac{[\hat{K}\,3]^3}{[3\,4][4\,5][5\,\hat{6}][\hat{6}\,\hat{K}]}$$

$$= -\frac{i}{s_{12}} \frac{[1\,2]^3}{([2\,\hat{K}]\langle\hat{K}\,6\rangle)(\langle 6\,\hat{K}\rangle[\hat{K}\,1])} \frac{(\langle 6\,\hat{K}\rangle[\hat{K}\,3])^3}{[3\,4][4\,5][5\,\hat{6}]([\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle)}$$

$$= i\,\frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]s_{612}\langle 2^-|(6+1)|5^-\rangle}$$

$$\langle 6\,\hat{K}\rangle[\hat{K}\,a] = \langle 6\,1\rangle\,[1\,a] + \langle 6\,2\rangle\,[2\,a] = \langle 6^-|(1+2)|a^-\rangle$$

$$[5\,\hat{6}] = [5\,6] + \frac{s_{12}[5\,1]}{\langle 6\,2\rangle[2\,1]} = \frac{\langle 5^+|(6+1)|2^+\rangle}{\langle 6\,2\rangle}$$

$$[\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle = \langle 6^+|(1+2)|6^+\rangle + s_{12} = s_{612}$$

Simple final form

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,s_{612}\langle 2^-|(6+1)|5^-\rangle} + \frac{\langle 4^-|(5+6)|1^-\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,1]\,s_{561}\langle 2^-|(6+1)|5^-\rangle}$$

Simpler than form found in 1980s — Mangano, Parke, Xu (1988)

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{([1\,2]\,\langle 4\,5\rangle\,\langle 6^-|(1+2)|3^-\rangle)^2}{s_{61}s_{12}s_{34}s_{45}s_{612}} + \frac{([2\,3]\,\langle 5\,6\rangle\,\langle 4^-|(2+3)|1^-\rangle)^2}{s_{23}s_{34}s_{56}s_{61}s_{561}} + \frac{s_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6^-|(1+2)|3^-\rangle\langle 4^-|(2+3)|1^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

Conclusions

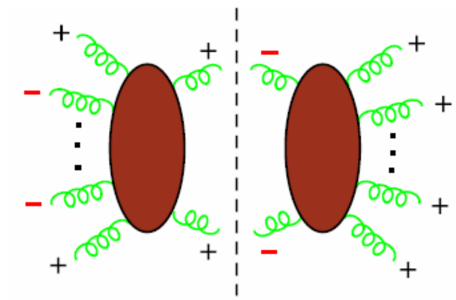
- NNLO progress needed for precision QCD
- Relies on computational advances; simple, few-scale processes first attacked analytically
- Flexible numerical approaches now approaching fruition
- Variety of results should be available in next few years
- More efficient calculation of QCD trees & loops in last year, related to twistor space perspective
- Practical spinoffs so far mostly for trees, and loops in supersymmetric theories
- Loop-level versions of on-shell recursion relations promising
- Expect much future progress along these lines

Extra slides

Motivation

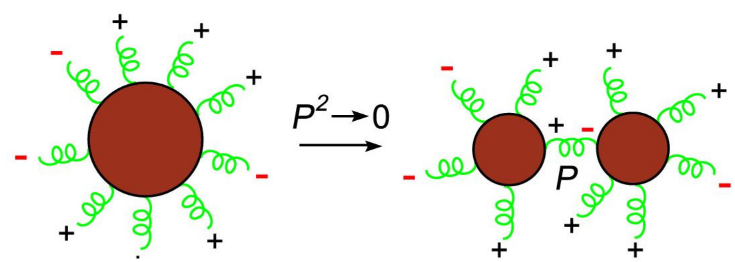
- What are the basic analytic properties of scattering amplitudes?

• Branch cuts

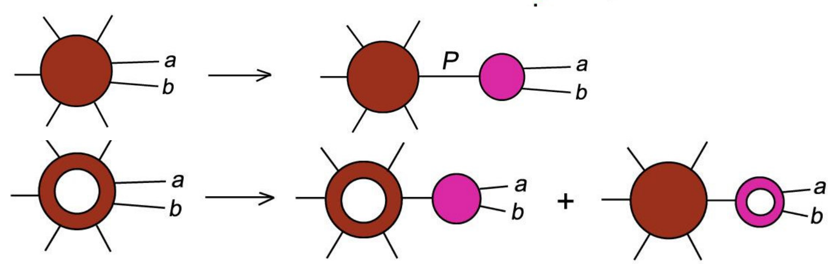


• Poles

- multi-particle



- collinear



Can we reconstruct scattering amplitudes directly from this information?

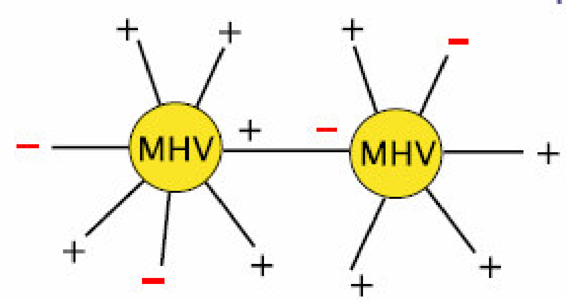
MHV rules

Cachazo, Svrček, Witten (2004)

- Continue MHV amplitudes off-shell:

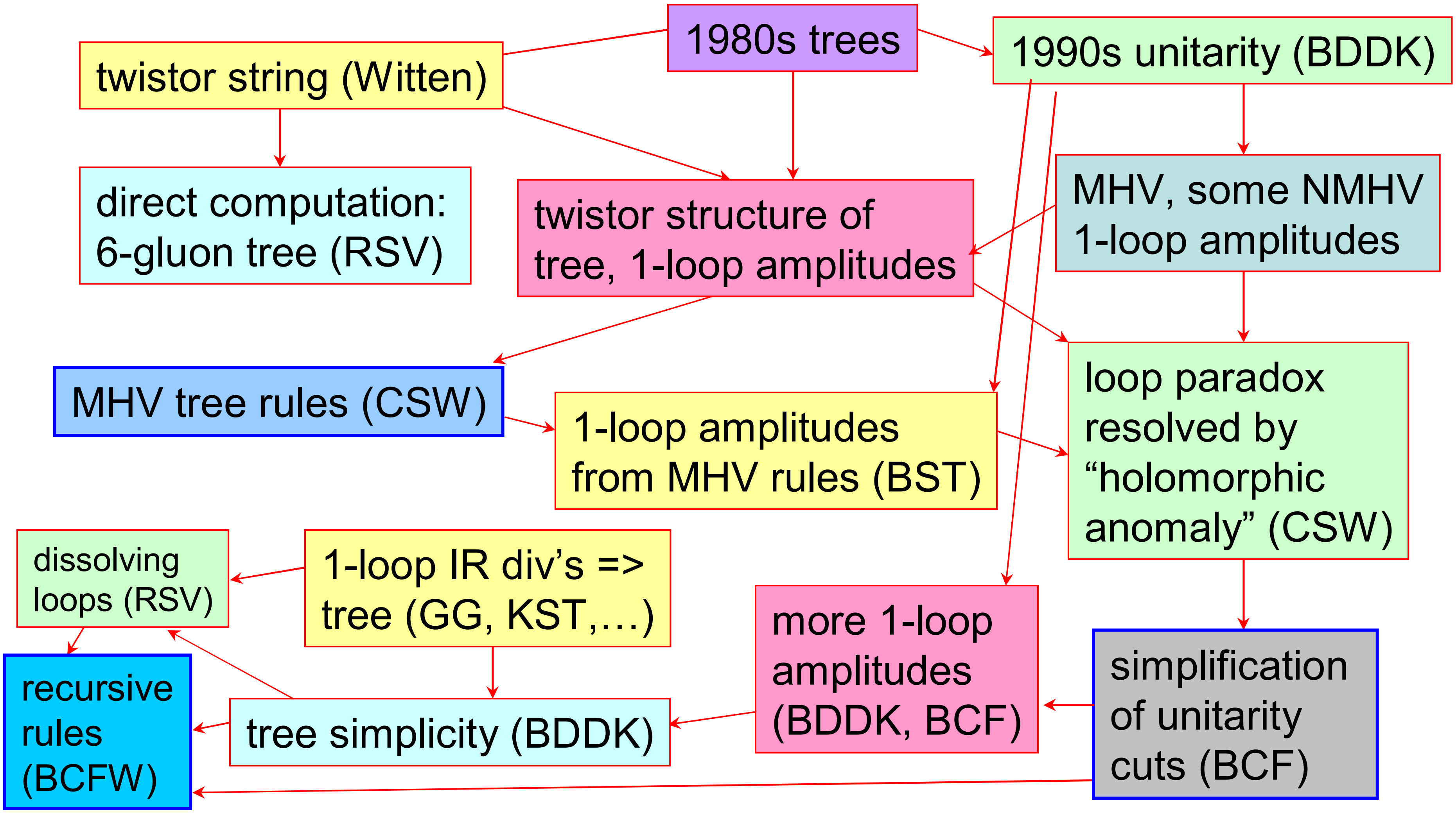
$$A_n^{\text{tree,MHV},ij}(1^*) = \frac{\langle i\, j\rangle^4}{\langle 1^*\, 2\rangle \ldots \langle n\, 1^*\rangle} = \frac{\langle i\, j\rangle^4}{\langle \eta^+ |\, 1\, |2^+\rangle \ldots \langle n^- |\, 1\, |\eta^-\rangle}$$

η is null, $\eta^2 = 0$. Sew vertices with "scalar" propagators. $\frac{1}{p^2}$



- Results independent of η, agree (numerically) with Feynman.

Where did recursive rules come from?



Recursion at one loop

Bern, LD, Kosower, hep-th/0501240

For simplicity, we considered special one-loop amplitudes with no cuts, only poles: $A_n^{1-\text{loop}}(1^\pm, 2^+, 3^+, \ldots, n^+)$

Still not quite a trivial extension of BCF because of:

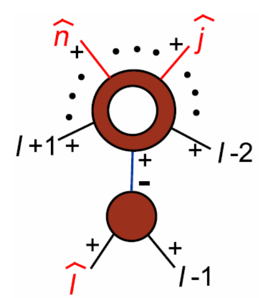
- difficulties with $z \to \infty$ for $A_n^{1-\text{loop}}(1^+, 2^+, 3^+, \ldots, n^+)$

$\Rightarrow$ need triple-shift

$$\hat{\tilde{\lambda}}_j = \tilde{\lambda}_j - z\tilde{\lambda}_l - z\frac{\langle n\, j\rangle}{\langle l\, j\rangle}\tilde{\lambda}_n$$

$$\hat{\lambda}_l = \lambda_l + z\lambda_j$$

$$\hat{\lambda}_n = \lambda_n + z\frac{\langle n\, j\rangle}{\langle l\, j\rangle}\lambda_j$$

$\Rightarrow$ 4-term recursion relation $\sim$ 

$+ \cdots$

Recursion at one loop (cont.)

- double pole for one z_k in $A_n^{1-\text{loop}}(1^-, 2^+, 3^+, \ldots, n^+)$

  $\Rightarrow$ need ansatz for single pole underneath

  $\Rightarrow$ $(n-2)$-term recursion relation

Agrees with (off-shell) recursive result — Mahlon, hep-ph/9312276

but expressions much simpler

$$
\begin{aligned}
&A_{6;1}(1^-,2^+,3^+,4^+,5^+,6^+) \\
&= i\,\frac{N_p}{96\pi^2}\Bigg[\frac{\langle 1^-|(2+3)|6^-\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 4\,5\rangle^2 s_{123}\,\langle 3^-|(1+2)|6^-\rangle} + \frac{\langle 1^-|(3+4)|2^-\rangle^3}{\langle 3\,4\rangle^2\langle 5\,6\rangle\langle 6\,1\rangle\, s_{234}\,\langle 5^-|(3+4)|2^-\rangle} \\
&\quad + \frac{[2\,6]^3}{[1\,2]\,[6\,1]\, s_{345}}\left(\frac{[2\,3]\,[3\,4]}{\langle 4\,5\rangle\,\langle 5^-|(3+4)|2^-\rangle} - \frac{[4\,5]\,[5\,6]}{\langle 3\,4\rangle\,\langle 3^-|(1+2)|6^-\rangle} + \frac{[3\,5]}{\langle 3\,4\rangle\langle 4\,5\rangle}\right) \\
&\quad - \frac{\langle 1\,3\rangle^3\,[2\,3]\,\langle 2\,4\rangle}{\langle 2\,3\rangle^2\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,1\rangle} + \frac{\langle 1\,5\rangle^3\,\langle 4\,6\rangle\,[5\,6]}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle^2\langle 5\,6\rangle^2} \\
&\quad - \frac{\langle 1\,4\rangle^3\,\langle 3\,5\rangle\,\langle 1^-|(2+3)|4^-\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle^2\langle 4\,5\rangle^2\langle 5\,6\rangle\langle 6\,1\rangle}\Bigg]
\end{aligned}
$$

A few “twistor” references

- Topological string in twistor space Witten, hep-th/0312171
- MHV (CSW) rules Cachazo, Svrcek, Witten, hep-th/0403047
- All 7-point N=4 1-loop amplitudes
  Bern, Del Duca, LD, Kosower hep-th/0410224
- Generalized unitarity & N=4 1-loop amplitudes
  Britto, Cachazo, Feng, hep-th/0412103
- All n-point NMHV N=4 1-loop amplitudes Bern, LD, Kosower hep-th/0412210
- Dissolving loops into trees Roiban, Spradlin, Volovich, hep-th/0412265
- BCF rules Britto, Cachazo, Feng, hep-th/0412308
- BCFW proof Britto, Cachazo, Feng, Witten, hep-th/0501052
- Recursive rules at 1 loop Bern, LD, Kosower hep-th/0501240