Saturation, traveling waves, and the BK equation at LL and NLL

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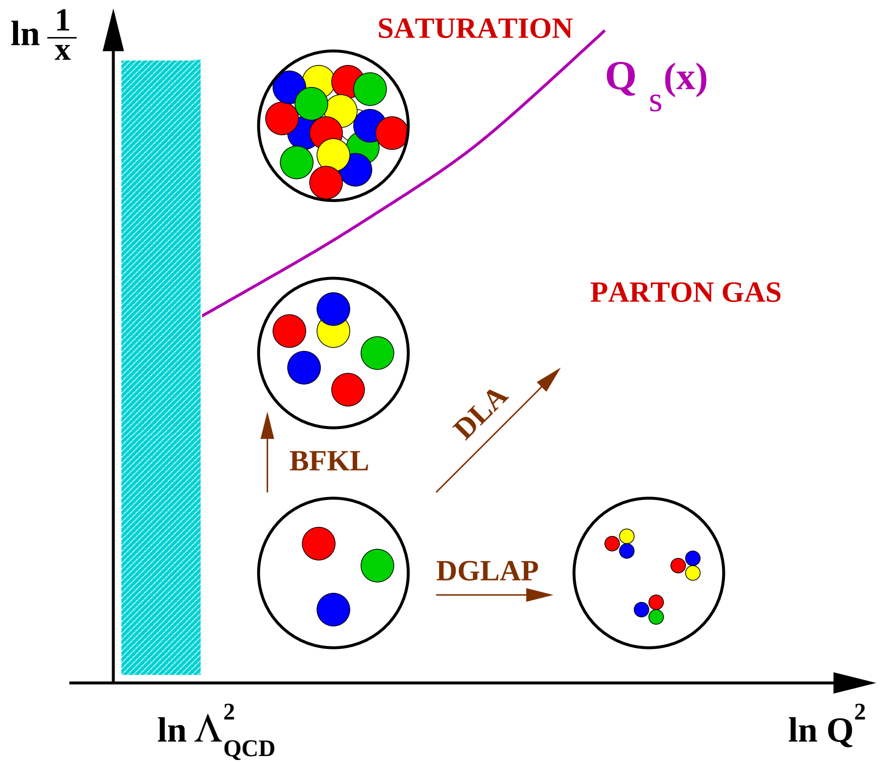
Overview

- About traveling waves

- Numerical studies of BK equation
- Fluctuations and stochastic evolution

  [RE, K. Golec-Biernat, S. Munier – in preparation]
- NLL corrections to traveling waves [RE – in preparation]

Introduction to saturation



- Gluon density grows when $Y \sim \log 1/x$ grows $\Rightarrow$
  - Gluon recombination possible
  - Alternatively, unitarization of dipole amplitude
- Reduction of cross section for very small x ?

Balitsky–Kovchegov equation

The BK equation in momentum space is

$$\partial_Y \mathcal{N}(k,Y) = \bar{\alpha}\left[K \otimes \mathcal{N}(k,Y) - \mathcal{N}^2(k,Y)\right]$$

where K is the integral kernel of the BFKL equation.

Formally,

$$\partial_Y \mathcal{N}(k,Y) = \bar{\alpha}\left[\chi(-\partial_L)\mathcal{N}(k,Y) - \mathcal{N}^2(k,Y)\right]$$

where

$$\begin{cases} L = \log(k^2/k_0^2) \\ \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \end{cases}$$
(BFKL kernel in Mellin space)

BK equation and traveling waves

- $\partial_Y \mathcal{N}(k, Y) = \bar{\alpha}\left[\chi(-\partial_L)\mathcal{N}(k, Y) - \mathcal{N}^2(k, Y)\right]$
is a quite complicated integro-differential equation.
- Munier & Peschanski realized that it can be approximated by expanding the kernel:

$$\chi(\gamma) \simeq \chi(\gamma_c) + \chi'(\gamma_c)(\gamma - \gamma_c) + \frac{1}{2}\chi''(\gamma_c)(\gamma - \gamma_c)^2$$

which brings the BK equation into the form of the Fisher–Kolmogorov–Petrovsky–Piscounov (FKPP) equation

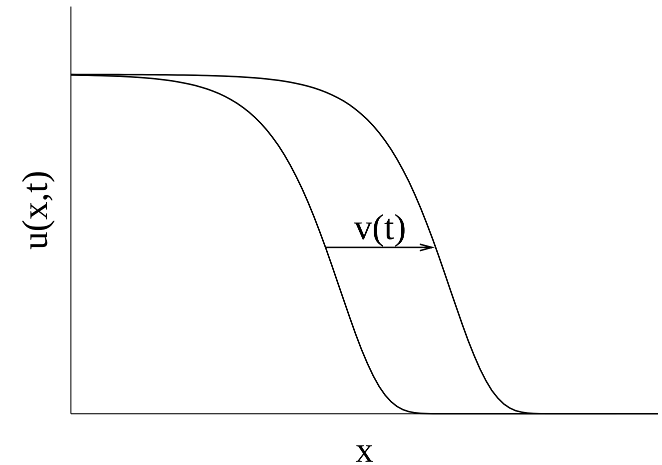
$$\partial_t u = \partial_x^2 u + u - u^2$$

($t \sim Y$ and $x \sim L = \log k^2$)
- FKPP equation has *traveling wave* solutions

Traveling wave fronts

Traveling waves are translating in space with a fixed shape,

$$u(x,t) = u(x - vt)$$



The position of the front, $x(t)$ corresponds to $\log Q_s^2(Y)$ for BK so geometric scaling comes natural:

$$\mathcal{N}(L,Y) = \mathcal{N}(\log k^2 - \log Q_s^2(Y)) = \mathcal{N}(\log(k^2/Q_s^2(Y)))$$

where $Q_s^2(Y)$ is the saturation scale.

Saturation scale

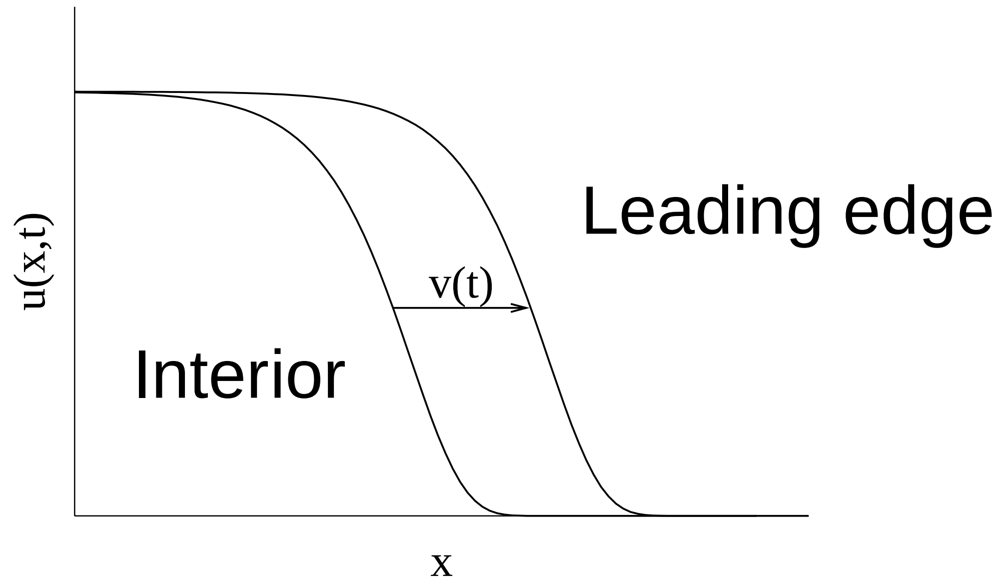
The *saturation scale* $Q_s^2(Y)$ is the momentum that separates the "saturated" region from the "linear" region. It is given by

$$\log Q_s^2(Y) = \bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c}Y - \frac{3}{2\gamma_c}\log Y - \frac{3}{\gamma_c^2}\sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}}\frac{1}{\sqrt{Y}} + \mathcal{O}(1/Y)$$

γ_c is the critical γ-value — not usual BFKL $\gamma_0 = 1/2$

Determined by $\boxed{\chi'(\gamma_c)\gamma_c = \chi(\gamma_c)} \Rightarrow \gamma_c = 0.627\ldots$

Solution for amplitude



The interior region depends on the nonlinearity.
The leading edge shape is given by the linearized equation.

$$\mathcal{N}(z,Y) \sim \log\left(\frac{k^2}{Q_s^2(Y)}\right)\left(\frac{k^2}{Q_s^2(Y)}\right)^{-\gamma_c}\exp\left(-\frac{\log^2\left(\frac{k^2}{Q_s^2(Y)}\right)}{2\bar{\alpha}\chi''(\gamma_c)Y}\right)$$

[Munier & Peschanski, hep-ph/0310357]

General method!

Important: this is much more general than FKPP

- No need to know the shape of the nonlinearity

  $\longrightarrow$ The nonlinearity selects the speed of the front, but this speed can be *calculated from the linearized equation*

  [See U. Ebert & W. van Saarloos, Physica D 146, 1 (2000) for more details]

- The equation *does not have to be a differential equation*

  $\longrightarrow$ enough to know dispersion relation in Mellin space!

- This is *not* incompatible with solutions in deep saturation region (interior region)!

  $\longrightarrow$ traveling wave method only gives front shape in leading edge region!

Numerical simulation

The traveling wave method gives nice results,
but how well can we trust it?

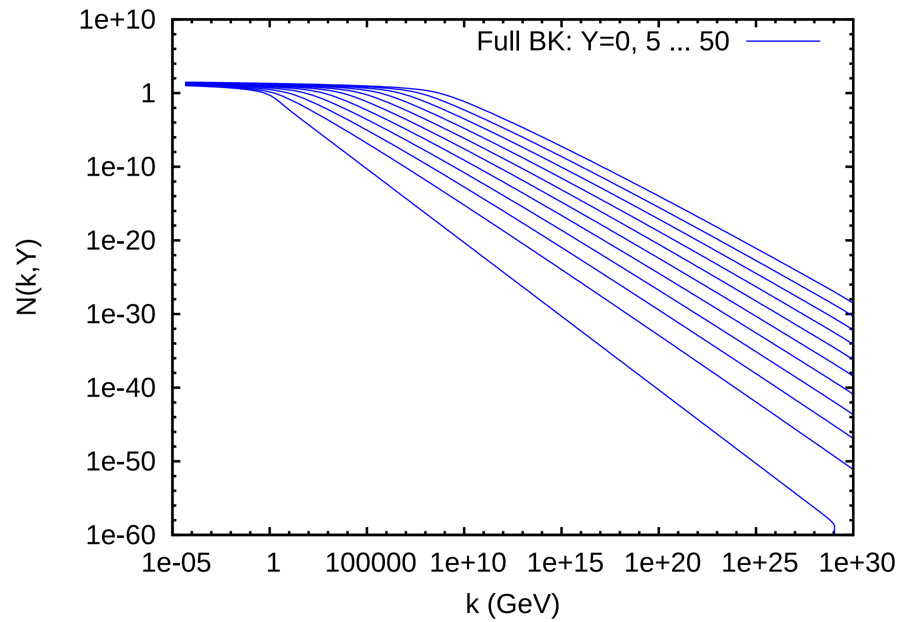
Solve BK numerically to check traveling wave results
[RE, K. Golec-Biernat, S. Munier, paper in preparation]

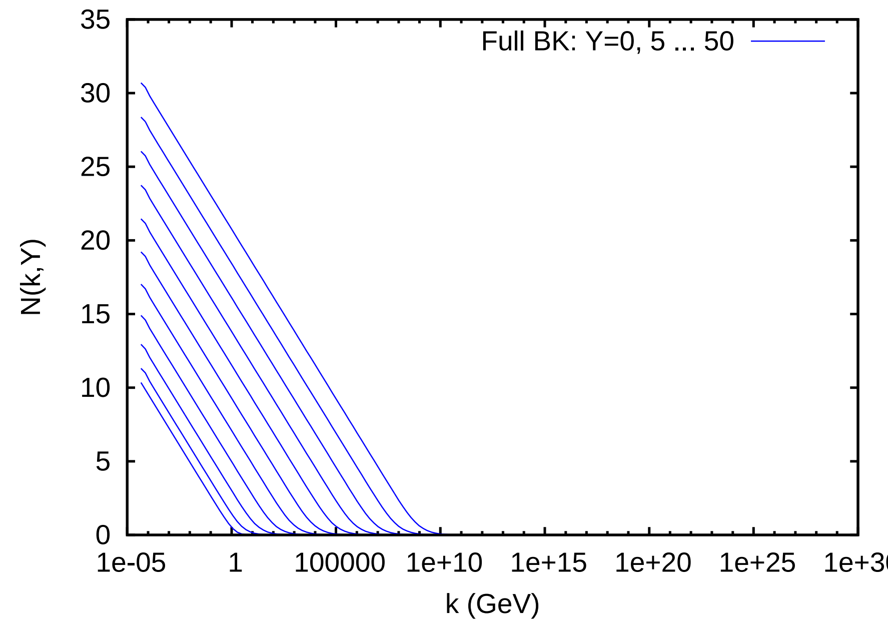
Numerically solve the equation

$$\partial_Y \frac{\mathcal{N}(k,Y)}{\bar{\alpha}} = \int \frac{dk'^2}{k'^2} \left[\frac{k'^2 \mathcal{N}(k',Y) - k^2 \mathcal{N}(k,Y)}{|k^2 - k'^2|} + \frac{k^2 \mathcal{N}(k,Y)}{\sqrt{4k'^4 + k^4}} \right]$$
$$- \mathcal{N}^2(k,Y)$$

Numerical results

For example, using the McLerran–Venugopalan (MV) model as initial condition, in rapidity steps of $\Delta Y = 5$:





We clearly have traveling wave front solutions and approximate geometrical scaling for large rapidities!

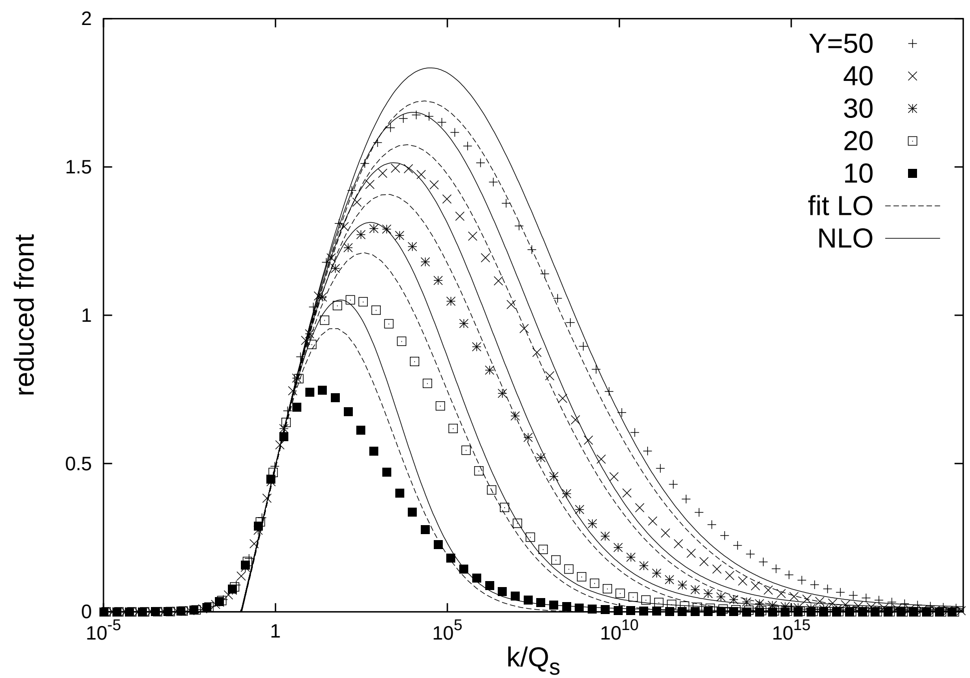
Front shape in leading edge region

What about geometrical scaling? — define reduced front

$$\mathcal{M}(k,Y) \equiv \mathcal{N}(k,Y) \times (k^2/Q_s^2(Y))^{\gamma_c}$$
$$= C_1 \left[\ln\left(\frac{k^2}{Q_s^2(Y)}\right) + C_2\right] \exp\left(-\frac{1}{4DY}\ln^2\left(\frac{k^2}{Q_s^2(Y)}\right)\right)$$

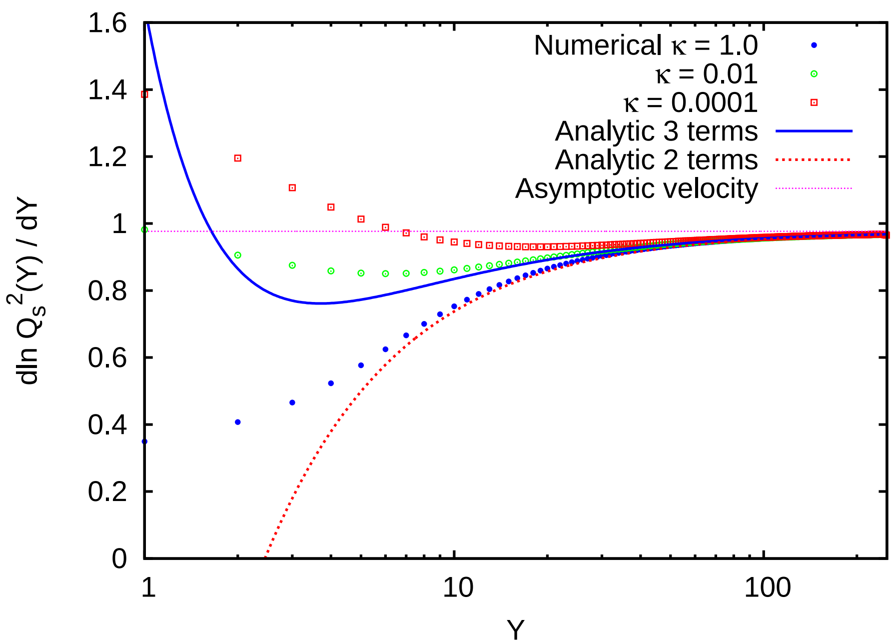
Selects the leading edge; shows diffusive scaling violations

Fit analytical
to numerical:



Saturation scale and subasymptotics

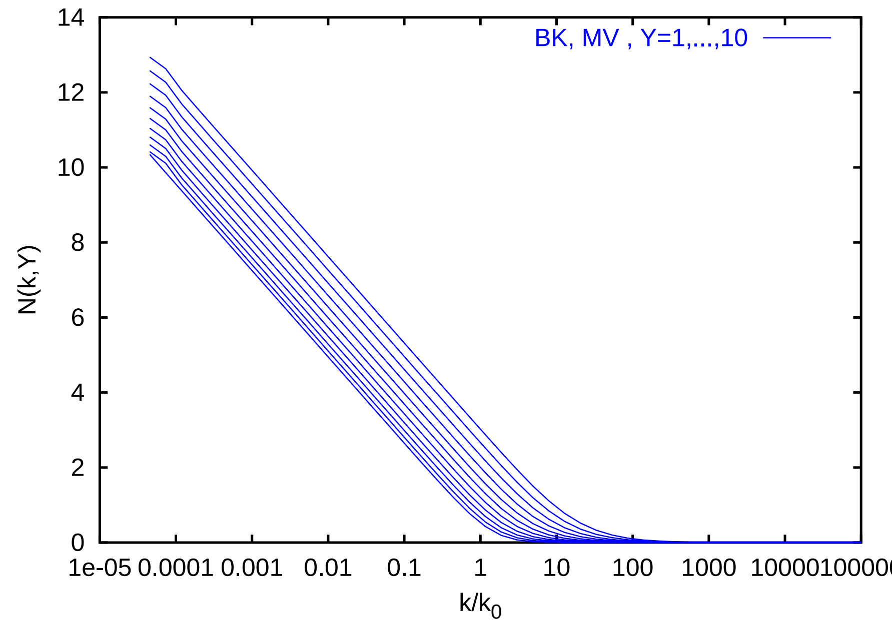
$$\partial \log Q_s^2(Y)/\partial Y$$



- Numerical solution agrees with analytic curves for large rapidities
- For smaller rapidities there are still subasymptotic terms
- $Q_s^2(Y)$ found by tracking point of fixed amplitude $\mathcal{N}(k, Y) = \kappa$ as Y increases

Subasymptotics and initial condition

McLerran–Venugopalan initial condition, curves for $Y = 0, 1, \ldots, 10$

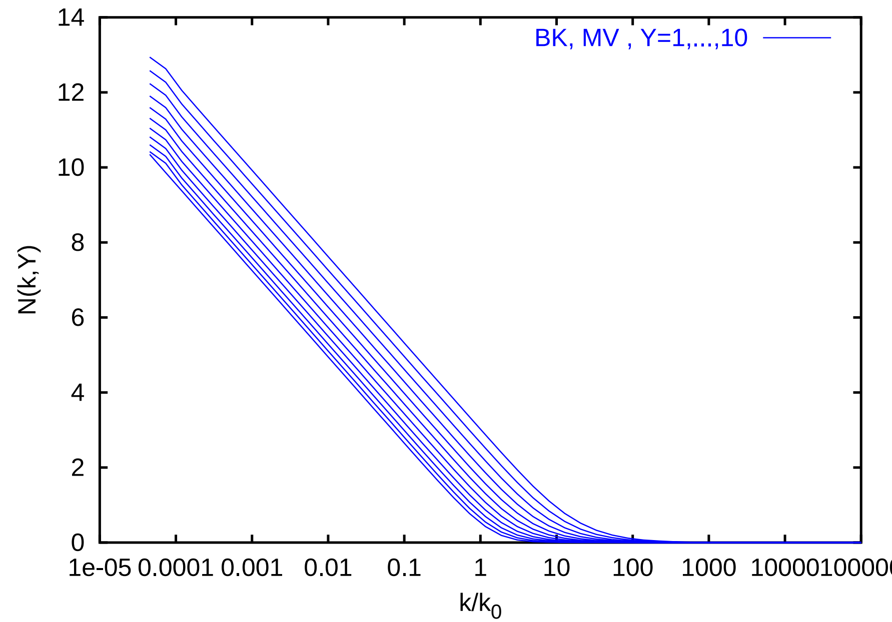


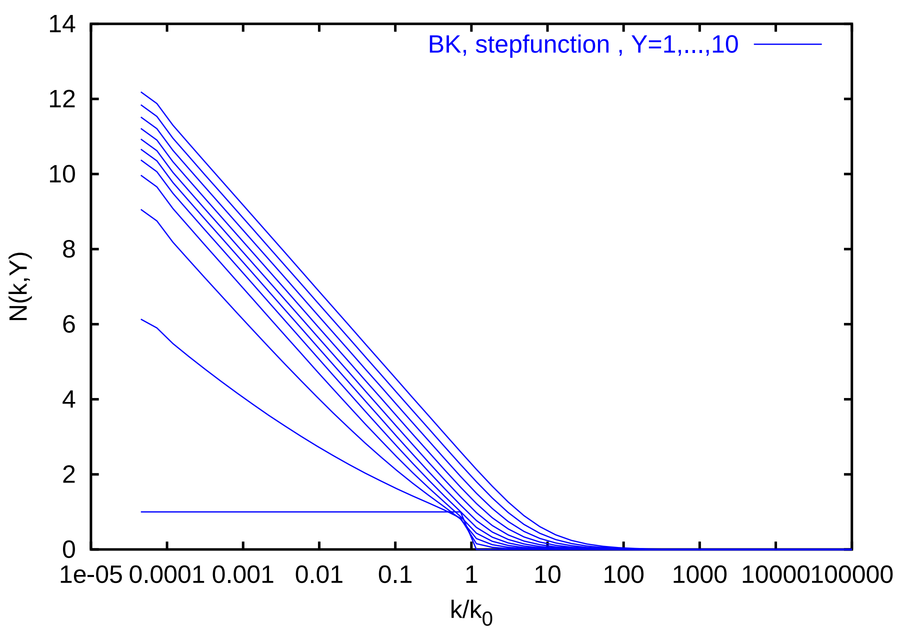
Subasymptotics and initial condition

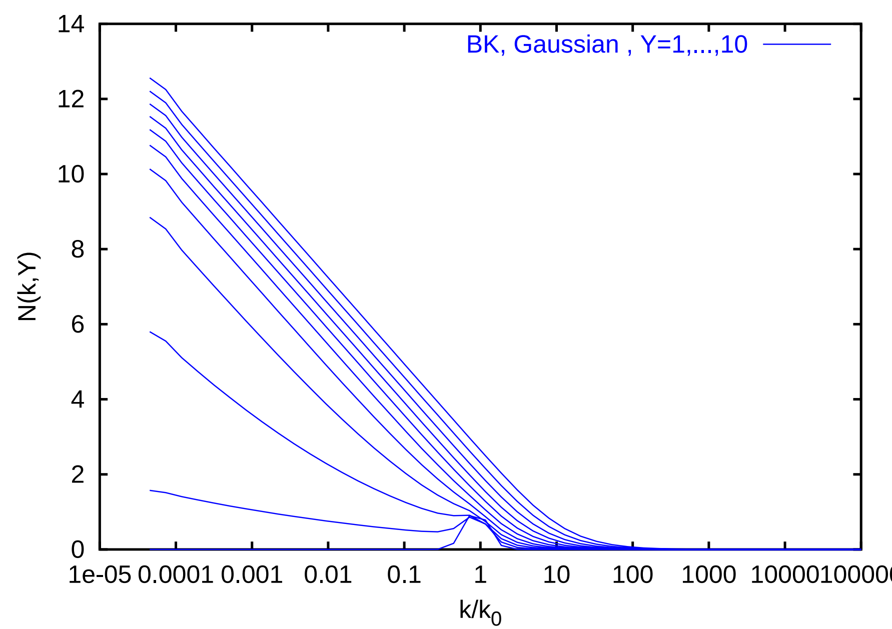
McLerran–Venugopalan initial condition, curves for $Y = 0, 1, \ldots, 10$

step function and Gaussian:

$\longrightarrow$ approach same velocity







Conclusions: numerical simulations

- The analytical results from traveling wave methods agree very well with the full numerical solution for large rapidities
- There are subleading terms in Y that depend on the initial conditions
- Everything I showed was for fixed coupling $\bar{\alpha}$
  — *running coupling works too...*
- We have not studied impact parameter dependence

Two modifications

- Fluctuations
- Next-to-leading corrections

Fluctuations and stochastic evolution

- The above BK picture is only a mean field picture
- It applies when the number of dipoles is large

$$\#\text{ dipoles} \equiv n = \mathcal{N}/\alpha_s^2$$

- When n is small, fluctuations become important
  n is discrete $\Rightarrow$ cutoff on amplitude!
- The asymptotic velocity becomes

$$\frac{d\ln Q_s^2(Y)}{dY} = \frac{\bar{\alpha}\chi(\gamma_c)}{\gamma_c} - \frac{\bar{\alpha}\pi^2}{2}\frac{\gamma_c\chi''(\gamma_c)}{\ln^2(1/\alpha_s^2)}$$

[Iancu, Mueller, Munier]

Fluctuations and stochastic evolution

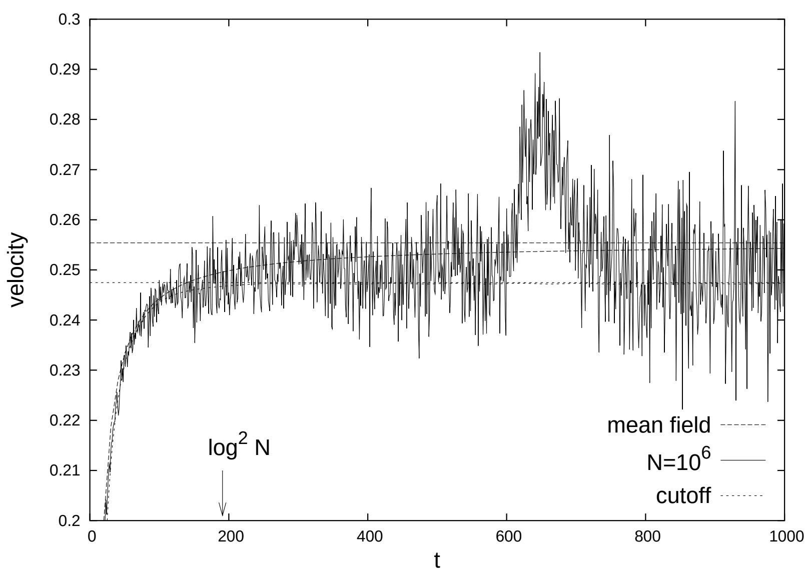
- Munier realized that the BK equation is in the equivalence class of the *stochastic* FKPP equation...

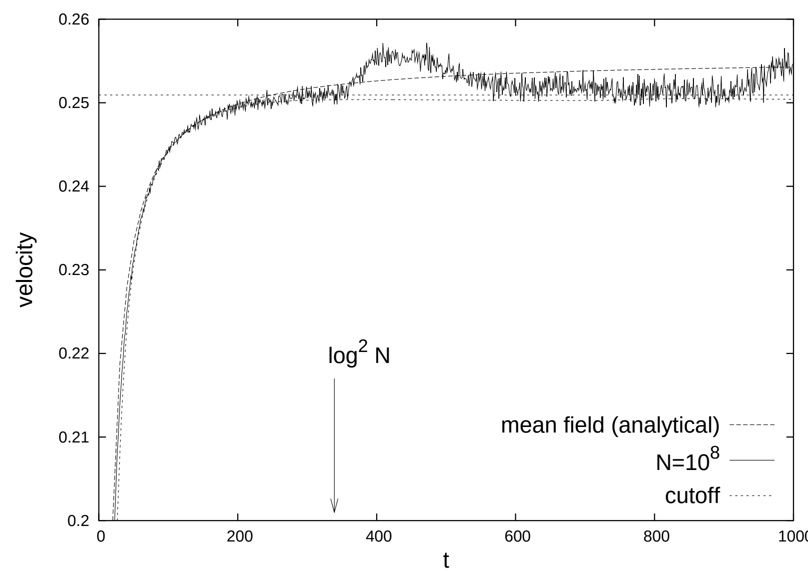
  Stochastic BK:

$$\partial_{\bar{\alpha}Y} T(k,Y) = \chi(-\partial_L) T(k,Y) - T^2(k,Y) + \alpha_s \sqrt{2T(k,Y)}\,\eta(k,Y)$$

- We study a toy model in the same equivalence class

[See also recent paper by G. Soyez]





NLL corrections to BK equation

- In particular Ciafaloni, Colferai and Salam have advocated using RG resummed BFKL kernels $\chi(\gamma, \omega)$ with an explicit ω-dependence.
- This leads to an NLL-corrected BK equation

$$\partial_Y \mathcal{N} = \bar{\alpha}\chi(-\partial_L, \partial_Y)\mathcal{N} - \bar{\alpha}\mathcal{N}^2$$

- NLL-corrections to nonlinear term neglected. But we don't need that term to compute the saturation scale!
- Generalize traveling wave method to this kind of equation [RE, in preparation]

NLL corrections to saturation scale

The saturation scale now takes the form

$$\ln Q_s^2(Y) = \frac{\omega_c}{\gamma_c} Y - \frac{3}{2\gamma_c} \ln Y - \frac{3}{\gamma_c^2} \sqrt{\frac{2\pi \chi_c' \gamma_c}{\omega_c Y}} \left(\chi_c'' + 2\,\frac{\omega_c}{\gamma_c}\,\dot{\chi}_c' + \left(\frac{\omega_c}{\gamma_c}\right)^2 \ddot{\chi}_c \right)^{-1/2}$$

where there are two critical constants (γ_c, ω_c) determined by the equations

$$\begin{cases} \chi'(\chi_c, \omega_c)\gamma_c = \chi(\gamma_c, \omega_c) - \omega_c\, \dot{\chi}(\gamma_c, \omega_c) \\ \omega_c = \bar{\alpha}\chi(\gamma_c, \omega_c) \end{cases}$$

which replace the LL relation $\chi_0'(\chi_c)\gamma_c = \chi_0(\gamma_c)$

RG resummed kernel

- The simplest NLL corrected kernel is the consistency constraint [Kwiecinski et al; Andersson et al] where the argument is shifted:

$$\chi(\gamma,\omega) = \psi(1) - \psi(\gamma + \tfrac{\omega}{2}) - \psi(1 - \gamma + \tfrac{\omega}{2}) \qquad \text{(symm)}$$

$$\chi(\gamma,\omega) = \psi(1) - \psi(\gamma) - \psi(1 - \gamma + \omega) \qquad \text{(asymm)}$$

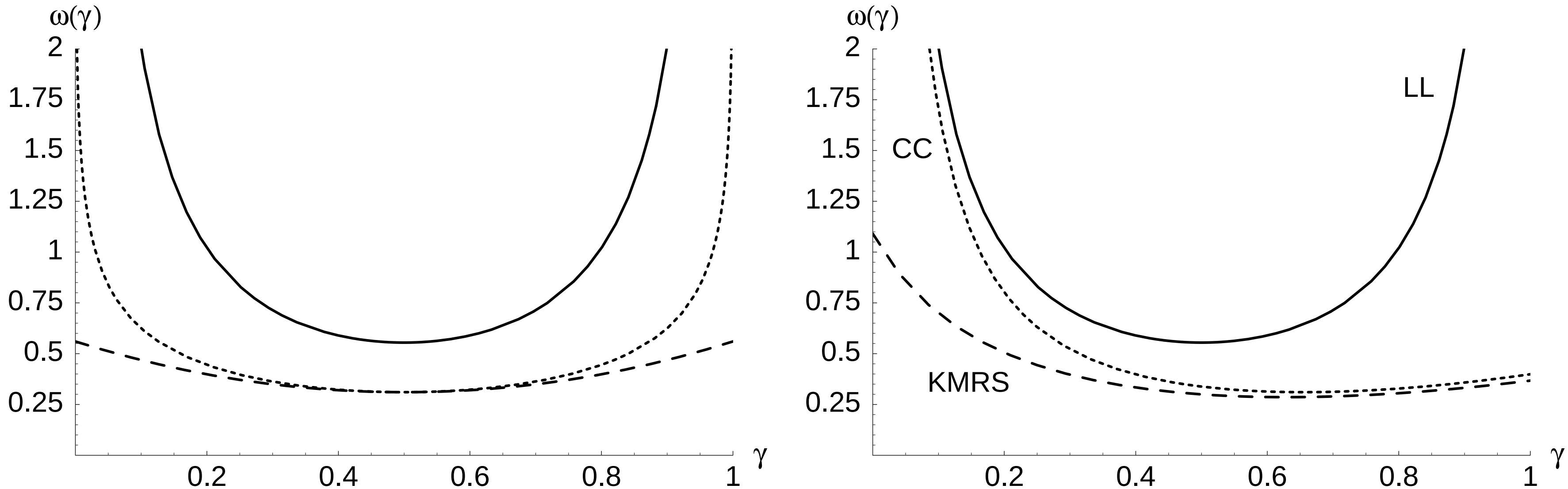
- Khoze, Martin, Ryskin and Stirling (KMRS) proposed a model similar to the NLL corrected LL BFKL kernel of Ciafaloni et al:

$$\chi(\gamma,\omega) = \chi_0(\gamma) + \frac{1+\omega A_1(\omega)}{\gamma} - \frac{1}{\gamma} + \frac{1+\omega A_1(\omega)}{1-\gamma+\omega} - \frac{1}{1-\gamma} - \omega\chi_0^{\text{ht}}(\gamma$$

Let's compare these, and the rapidity veto. . .

RG resummed kernels

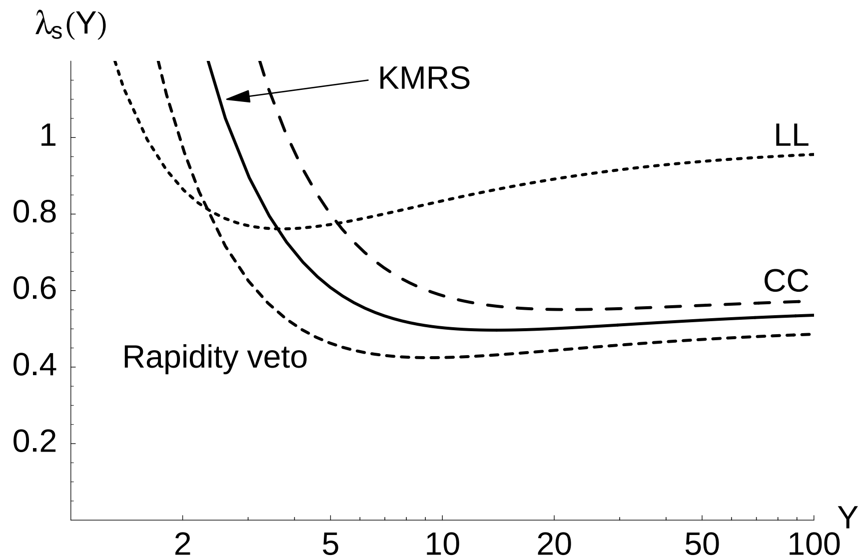
$\omega = \bar{\alpha}\chi(\gamma, \omega)$ defines $\omega(\gamma)$ implicitly — solve numerically:



- The modified kernels obviously have smaller pomeron intercepts [$\alpha_{\mathbb{P}} = \omega(\frac{1}{2})$], and they are also wider $\rightarrow$ smaller diffusion constant
- What about the saturation scale?

Saturation scale

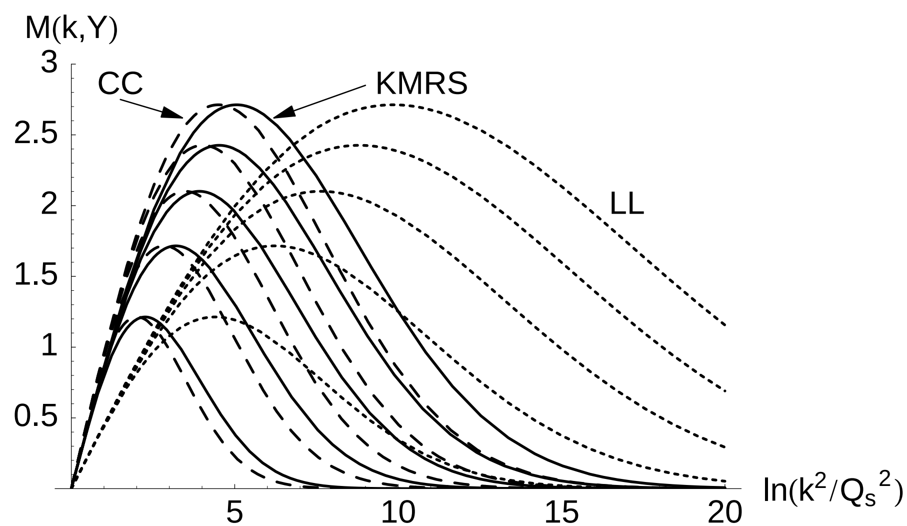
Logaritmic derivative of the saturation scale $\lambda_s(Y) \equiv \dfrac{d \ln Q_s^2}{dY}$



- Again, asymptotically constant, but not for small Y!
- The NLL corrections suppress the leading exponential growth of the saturation scale but also changes the approach to asymptotics

Reduced wave front

$$\mathcal{M}(k,Y) \equiv \mathcal{N}(k/Q_s(Y),Y) \times (k^2/Q_s^2(Y))^{\gamma_c}$$
$$= \frac{1}{\sqrt{D}} \ln\left(\frac{k^2}{Q_s^2(Y)}\right) \exp\left(-\frac{1}{4DY}\ln^2\left(\frac{k^2}{Q_s^2(Y)}\right)\right)$$



Diffusion constant smaller for NLL kernels.

Conclusions

- The NLL corrections $\rightarrow$ kernel non-local in both Y and k $\rightarrow$ "rotates evolution"
- NLL corrections reduce the saturation scale and the diffusive spreading of the wave front
- It's easy to compute the NLL corrections for fixed coupling
- Running coupling is a little more difficult
- Work in progress