Mini-Review of selected areas of BFKL

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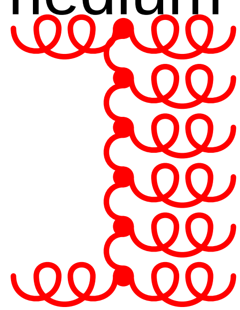
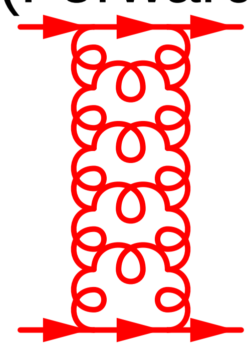
Outline of the talk

- The **High Energy Limit** of (diffractive) scattering processes and the **BFKL Resummation Framework**
- The solution of the **BFKL equation** at **Leading Logarithmic Accuracy**:
  - Eigen-values & E-functions
  - Iterative method
    What the iterative method gives extra
- Formalism at **Next to Leading Logarithmic Accuracy**
  - Why go to NLL for non-forward equation? It doesn't even work for $t = 0$GeV!?
  - ...but it does!
    Why one *has* to use a different approach at NLL
- **Non-forward** BFKL Equation at LLA (and progress towards NLLA) for **Diffractive processes**

BFKL formalism

- **BFKL** (**B**alitskii, **F**adin, **K**uraev, **L**ipatov): resummation of **large logarithms** in the perturbation series for QCD processes with **two large** (perturbative) and **disparate** energy scales $\hat{s} \gg |\hat{t}|$ ($\hat{s}$: E^2, $|\hat{t}|$: $p_\perp^2$)

Structure Functions	Forward Physics @ Hadron Colliders (Colour Octet Exchange)	Diffraction (Colour Singlet Exchange)
Small x	Large Rapidity (Forward) medium x 	Large Rapidity (Forward) 

BFKL formalism

- The cross section for the process $A + B \to A' + B'$ factorises as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{|A(s,t)|^2}{16\pi s^2},$$

$$\frac{|A(s,t)|}{s} = \left| \int \mathrm{d}^2\mathbf{k}_a \int \mathrm{d}^2\mathbf{k}_b \, \Phi_{\mathrm{A}}(\mathbf{k}_a, \mathbf{q}) \, \frac{f\left(\mathbf{k}_a, \mathbf{k}_b, \mathbf{q}, \Delta\right)}{(\mathbf{k}_a - \mathbf{q})^2 \mathbf{k}_b^2} \, \Phi_{\mathrm{B}}(\mathbf{k}_b, \mathbf{q}) \right|,$$

- $\Phi_A(\mathbf{k}_a, \mathbf{q}), \Phi_B(\mathbf{k}_b, \mathbf{q})$ process dependent *impact factors*
- $f\left(\mathbf{k}_a, \mathbf{k}_b, \mathbf{q}, \Delta\right)$ process independent *Gluon Green's function*

The BFKL Equation

The Gluon Green's function fulfil (to LLA and NLLA) the BFKL equation

$$\omega f_\omega\left(\mathbf{k}_a, \mathbf{k}_b, \mathbf{q}\right) = \delta^{(2)}\left(\mathbf{k}_a - \mathbf{k}_b\right) + \int d^2\mathbf{k}' \mathcal{K} \otimes f_\omega$$

where the BFKL kernel $\mathcal{K}\left(\mathbf{k}_a, \mathbf{k}', \mathbf{q}\right)$ is very recently calculated to NLLA.
Consider $t = 0$ for simplicity. Then at LL the kernel is **conformal invariant** with **eigenfunctions** $\mathbf{k}^{2(\gamma-1)}$. Use (transverse) Mellin transform!

$$\int d^2\mathbf{k}' \ \mathcal{K}\left(\mathbf{k}, \mathbf{k}'\right) \ \mathbf{k}^{2(\gamma-1)} = \frac{N_c \alpha_s}{\pi} \chi^{\mathrm{LL}}(\gamma) \mathbf{k}^{2(\gamma-1)}$$

$$\omega(\gamma) = \langle \gamma | \mathcal{K}(k, k) | \gamma \rangle$$

$$f_\omega(\mathbf{k}_a, \mathbf{k}_b) = \sum_\gamma \frac{\langle \gamma, \mathbf{k}_b | \gamma, \mathbf{k}_a \rangle}{\omega - \omega(\gamma)}$$

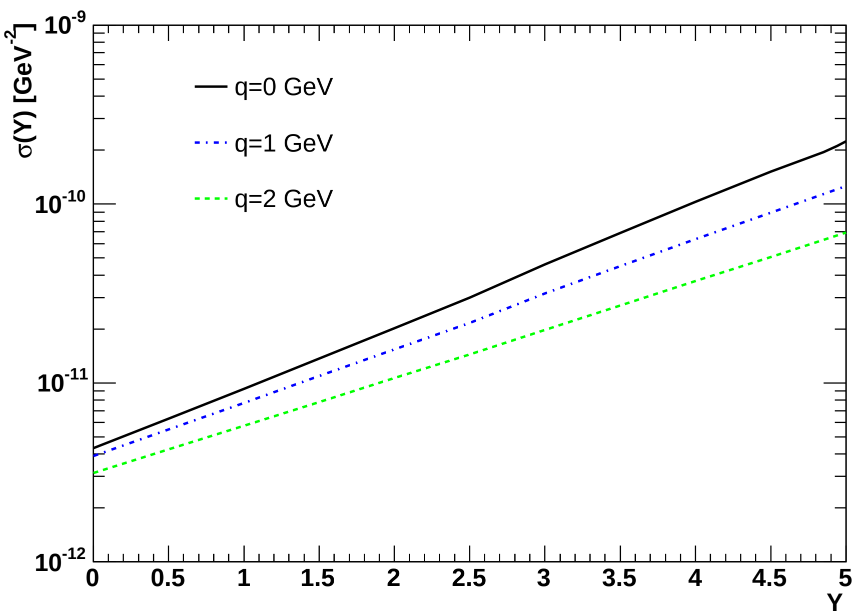
The BFKL Equation at LLA

Analytic solution for angular averaged gluon Green's function

$$\bar{f}\left(k_a, k_b, \mathbf{0}, \Delta\right) = \frac{1}{k_a k_b} \int_0^\infty d\nu \left(\frac{k_a^2}{k_b^2}\right)^{i\nu} e^{\bar{\alpha}_s \Delta \chi_0(\nu)}$$

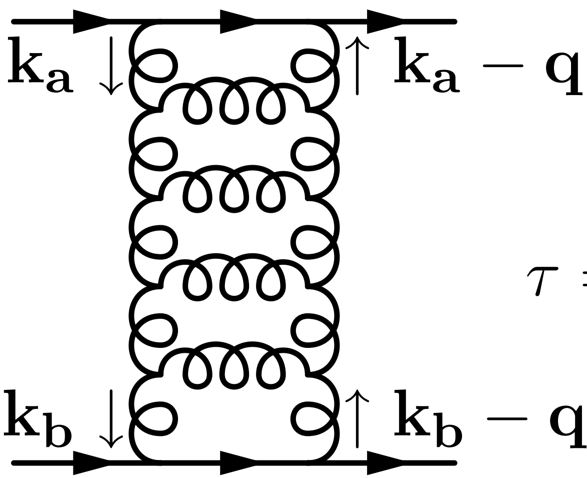
with the LL eigenvalue

$$\chi_0(\nu) = -2\,\mathrm{Re}\left\{\psi\left(\frac{1}{2}+i\nu\right)-\psi(1)\right\}.$$

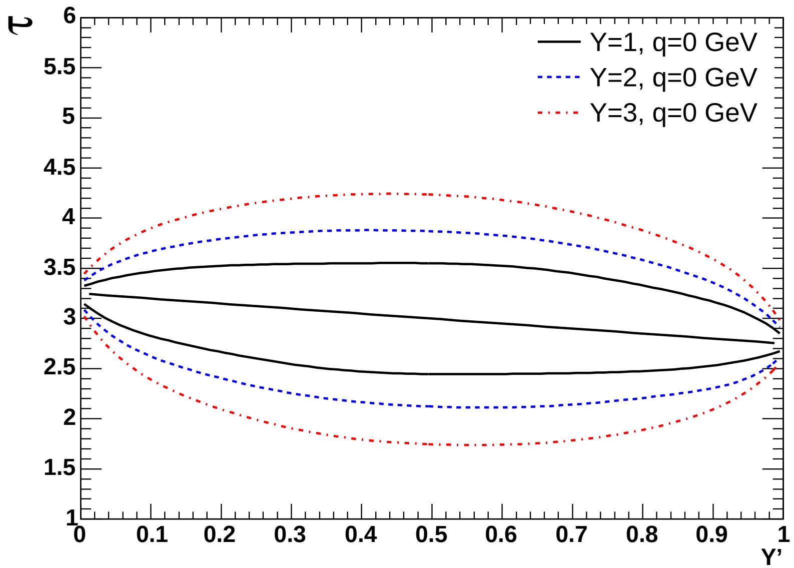


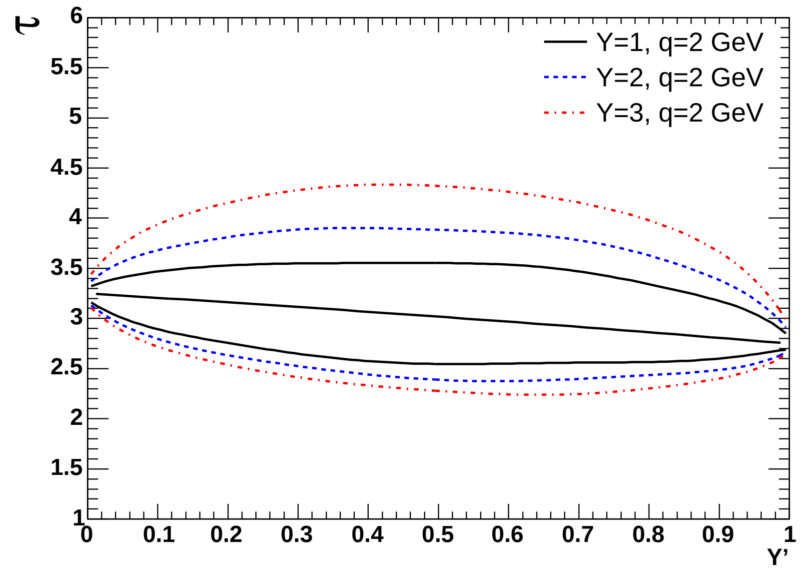
Exponential BFKL rise in cross section

Diffusion and Diffraction at LLA



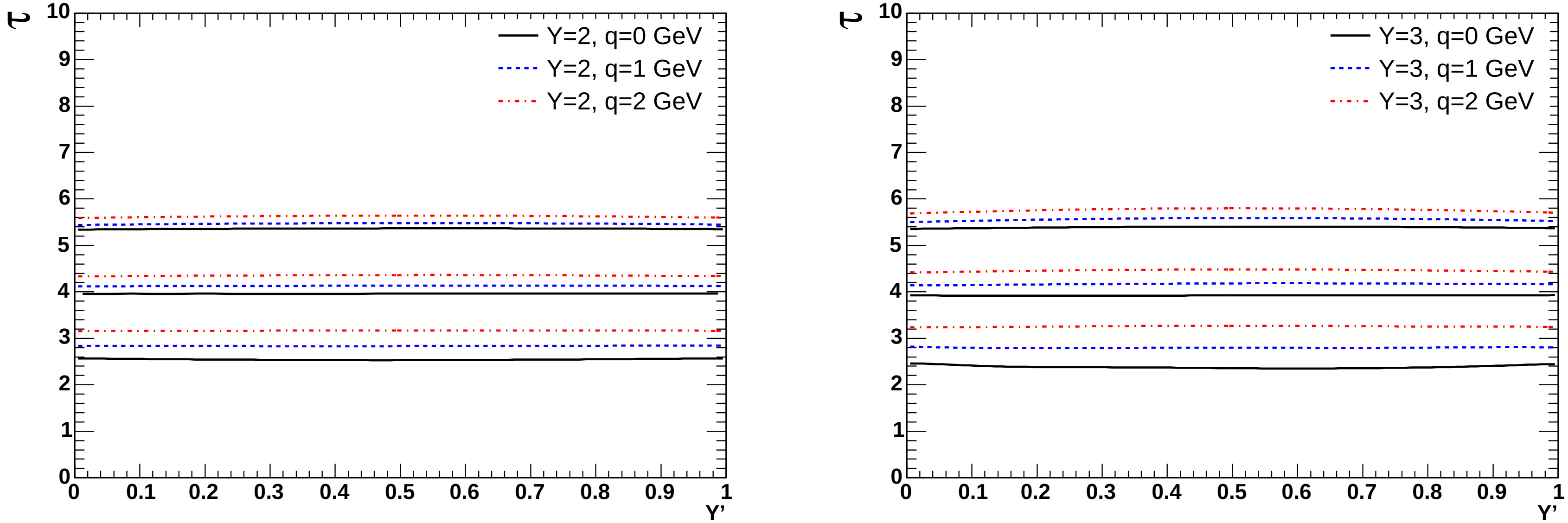
$$\tau = \ln\left(\left(k_a + \sum k_i\right)^2 / \mathrm{GeV}^2\right)$$





$k_a = 5$ GeV, $k_b = 4$ GeV

Diffusion for cross sections



The BFKL Equation at NLLA

- Both the *trajectory* $\omega(-\mathbf{k}_a^2)$ and the *real emission kernel* $\mathcal{K}_r$ are significantly more complicated than at LL
- Takes into account fermions and running coupling effects
- Furthermore, the impact factors at NLL are similarly complicated, and a fully analytic approach for cross sections seems almost hopeless
- We will propose a generalisation of the LL iterative solution that will solve the BFKL equation at NLL accuracy.
- But first the story so far…

The NLL BFKL Story So Far

- BFKL equation at LL put forward and solved in 1978.
  - **non-forward** equation solved five years later by L. Lipatov
- 8-10 years effort to calculate the BFKL kernel at NLLA ended in 1998
  - Initial results were discouraging. NLL kernel applied to LL eigenfunctions lead to huge and unstable corrections.
  - We will see why this analysis is invalid.
- Calculation of the **non-forward** kernel finished Dec. 2004 by V. Fadin and collaborators.

BFKL at NLLA

- Two new effects appear:
  - Fermions
  - Running Coupling

Conformal invariance **broken** (in QCD for $\beta \neq 0$) — Eigenfunctions **unknown**. Analyse what happens **if we pretend** the LL eigenfunctions are also eigenfunctions at NLL.

$$\omega^{\mathrm{NLL}}(\gamma) = \langle \gamma, \mathbf{k}_b | \mathcal{K}^{\mathrm{NLL}}(\mathbf{k}_b, \mathbf{k}_a) | \gamma, \mathbf{k}_a \rangle$$

$$f_\omega(\mathbf{k}_a, \mathbf{k}_b) = \sum_\gamma \frac{\langle \gamma, \mathbf{k}_b | \gamma, \mathbf{k}_a \rangle}{\omega - \omega^{\mathrm{NLL}}(\gamma)}$$

Leading Log tools at NLL

$$\omega^{\mathrm{NLL}}(\gamma) = \int \mathrm{d}^{D-2}\mathbf{k}\, \mathcal{K}^{\mathrm{NLL}}(\mathbf{k}_a, \mathbf{k}) \left(\frac{\mathbf{k}^2}{\mathbf{k}_a^2}\right)^{\gamma-1}$$

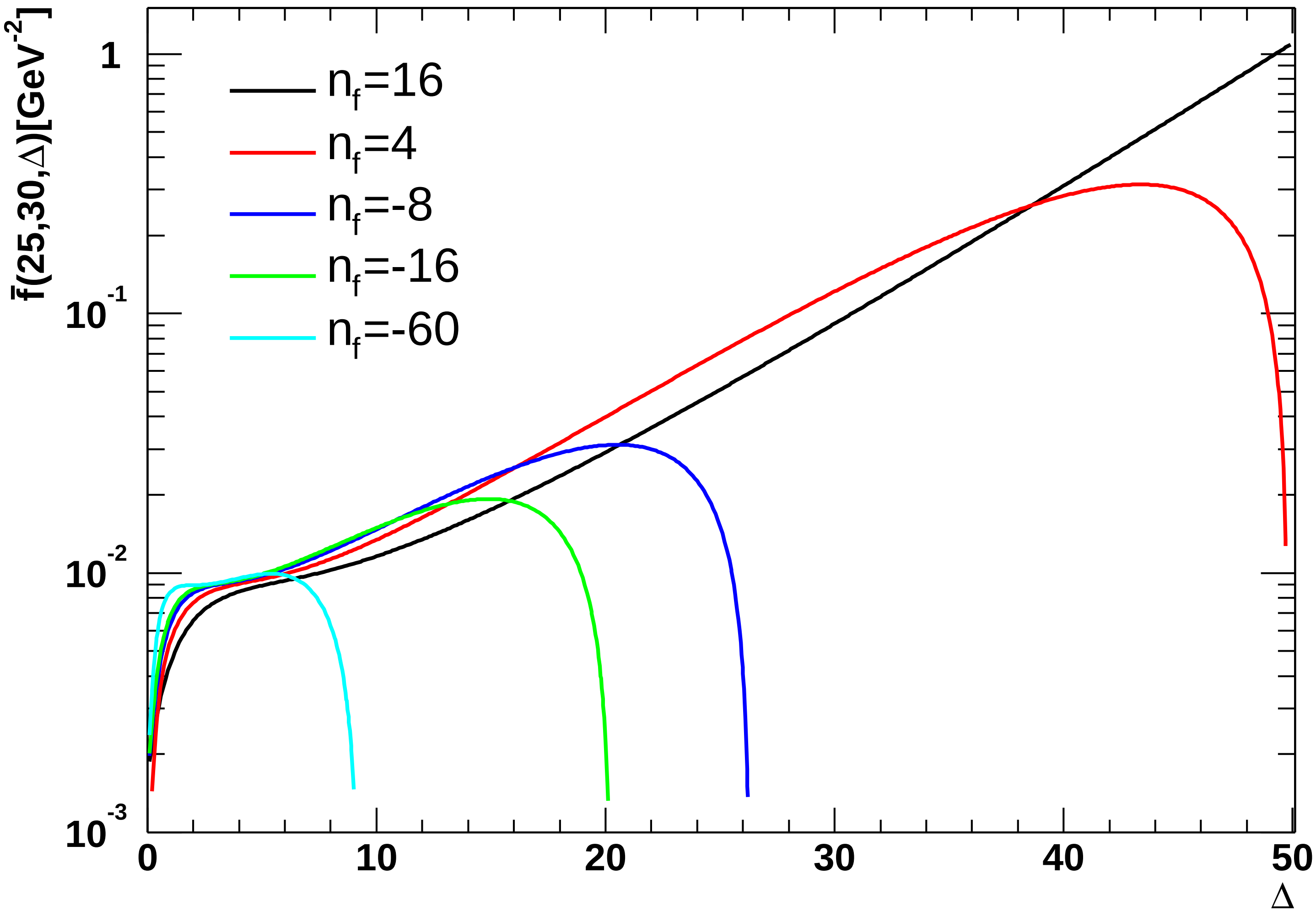
$$= \frac{\alpha_s(\mathbf{k}_a^2)N}{\pi}\left(\chi^{\mathrm{LL}}(\gamma) + \chi^{\mathrm{NLL}}(\gamma)\frac{\alpha_s(\mathbf{k}_a^2)N}{\pi}\right)$$

$$\chi^{\mathrm{NLL}}(\gamma) = -\frac{1}{4}\Big[\left(\frac{11}{3} - \frac{2}{3}\frac{n_f}{N}\right)\frac{1}{2}\left(\chi^{\mathrm{LL}}(\gamma) {\color{red}-\psi'(\gamma) + \psi'(1-\gamma)}\right)$$
$$- 6\zeta(3) + \frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left(3 + \left(1 + \frac{n_f}{N^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)$$
$$- \left(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{n_f}{N}\right)\chi^{\mathrm{LL}}(\gamma) - \psi''(\gamma) - \psi''(1-\gamma) - \frac{\pi^3}{\sin(\pi\gamma)} + 4\phi(\gamma)\Big],$$

let us pretend:

$$\bar{f}(k_a, k_b, \Delta) = \frac{1}{k_b^2}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\mathrm{d}\gamma}{2\pi i}\, e^{\Delta\omega^{\mathrm{NLL}}(\gamma)}\left(\frac{k_b^2}{k_a^2}\right)^{\gamma}$$

Leading Log tools at NLL

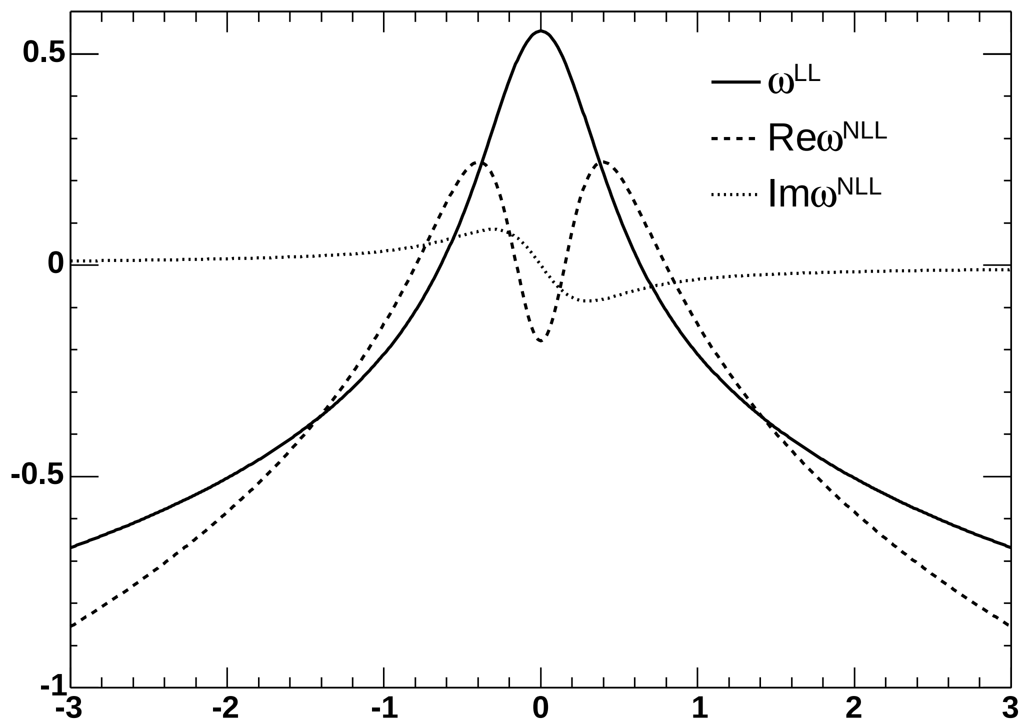


!!this would be a major catastrophe!!

Leading Log tools at NLL

$$\bar{f}(k_a, k_b, \Delta) = \frac{1}{k_b^2} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\mathrm{d}\gamma}{2\pi i}\, e^{\Delta\omega^{\mathrm{NLL}}(\gamma)} \left(\frac{k_b^2}{k_a^2}\right)^{\gamma}$$

$$\gamma = \frac{1}{2} + i\nu$$



Read the small print

Fadin and Lipatov say

Almost all the terms in the right hand side of eq. (12) except the contribution

$$\Delta(\gamma) = \frac{\alpha_s^2(\mu^2) N_c^2}{4\pi^2} \left(\frac{11}{3} - \frac{2n_f}{3N_c} \right) \frac{1}{2} \left(\psi'(\gamma) - \psi'(1-\gamma) \right)$$

are symmetric to the transformation $\gamma \leftrightarrow 1-\gamma$. Moreover, it is possible to cancel $\Delta(\gamma)$ if one would redefine the function $q^{2(\gamma-1)}$ by including in it the logarithmic factor $\left(\frac{\alpha_s(q^2)}{\alpha_s(\mu^2)}\right)^{-1/2}$.

This would **remove the imaginary part**, and therefore also **remove the oscillations**.

What to believe?

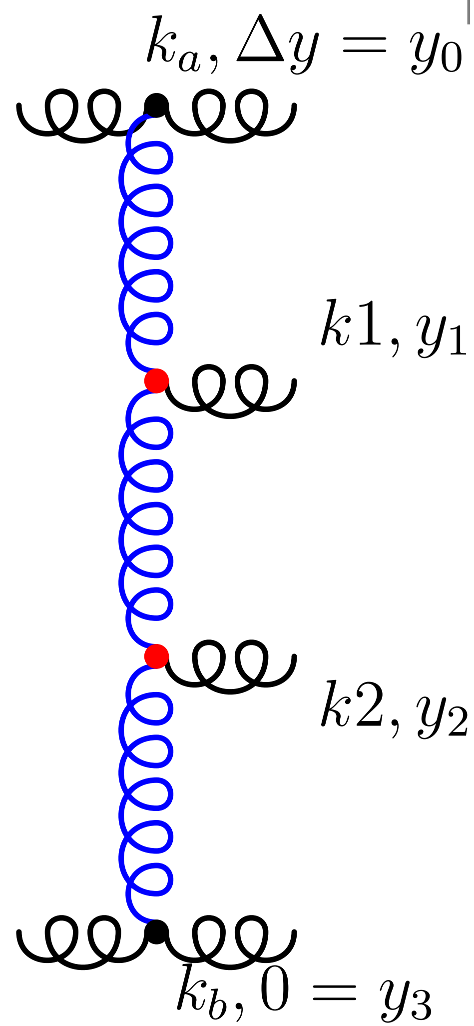
Iterative Solution at NLLA

We propose an iterative approach to the BFKL equation at NLLA that solves the equation with *no approximations*

- Directly in the physical rapidity and transverse momentum space
  (avoids the use of the troublesome Mellin transform completely)
- The right language for use of impact factors (physics predictions!)
- Expresses the solution in terms of effective vertices and no-emission probabilities (physical insight into the BFKL solution at NLLA!)

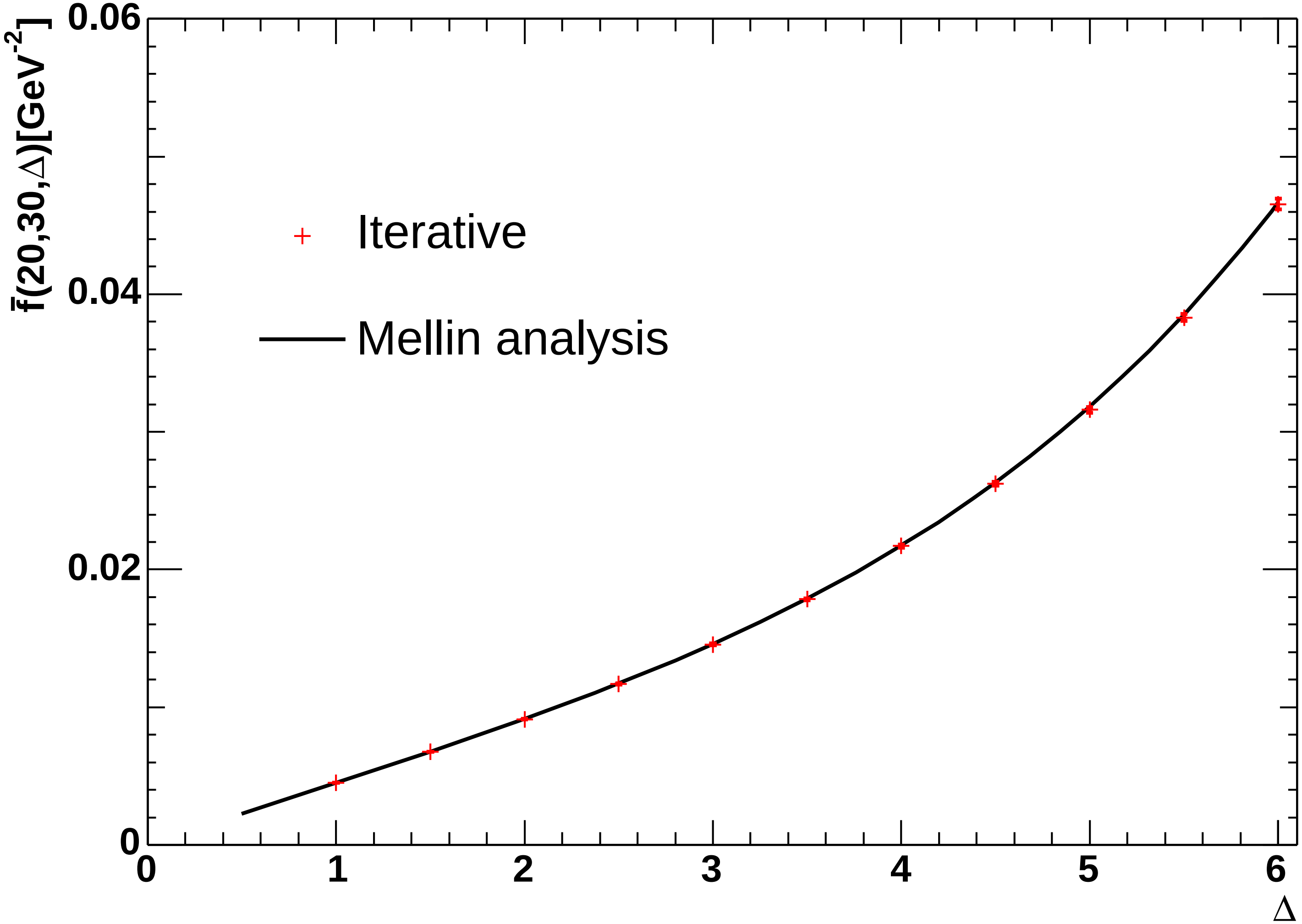
Enter Iteration at NLLA

$$
\begin{aligned}
f(\mathbf{k}_a, \mathbf{k}_b, \Delta) \;=\; & \exp\left(\omega_0\left(\mathbf{k}_a^2, \lambda^2, \mu\right)\Delta\right)\delta^{(2)}(\mathbf{k}_a - \mathbf{k}_b) \\
+ \sum_{n=1}^{\infty}\prod_{i=1}^{n}\int d^2\mathbf{k}_i \int_0^{y_{i-1}} dy_i & \left[V\left(\mathbf{k}_i, \mathbf{k}_a + \sum_{l=0}^{i-1}\mathbf{k}_l, \mu\right)\right] \\
\times \exp & \left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{i-1}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_{i-1} - y_i)\right] \\
& \times \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{n}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_n - 0)\right] \\
& \times \delta^{(2)}\left(\sum_{l=1}^{n}\mathbf{k}_l + \mathbf{k}_a - \mathbf{k}_b\right)
\end{aligned}
$$



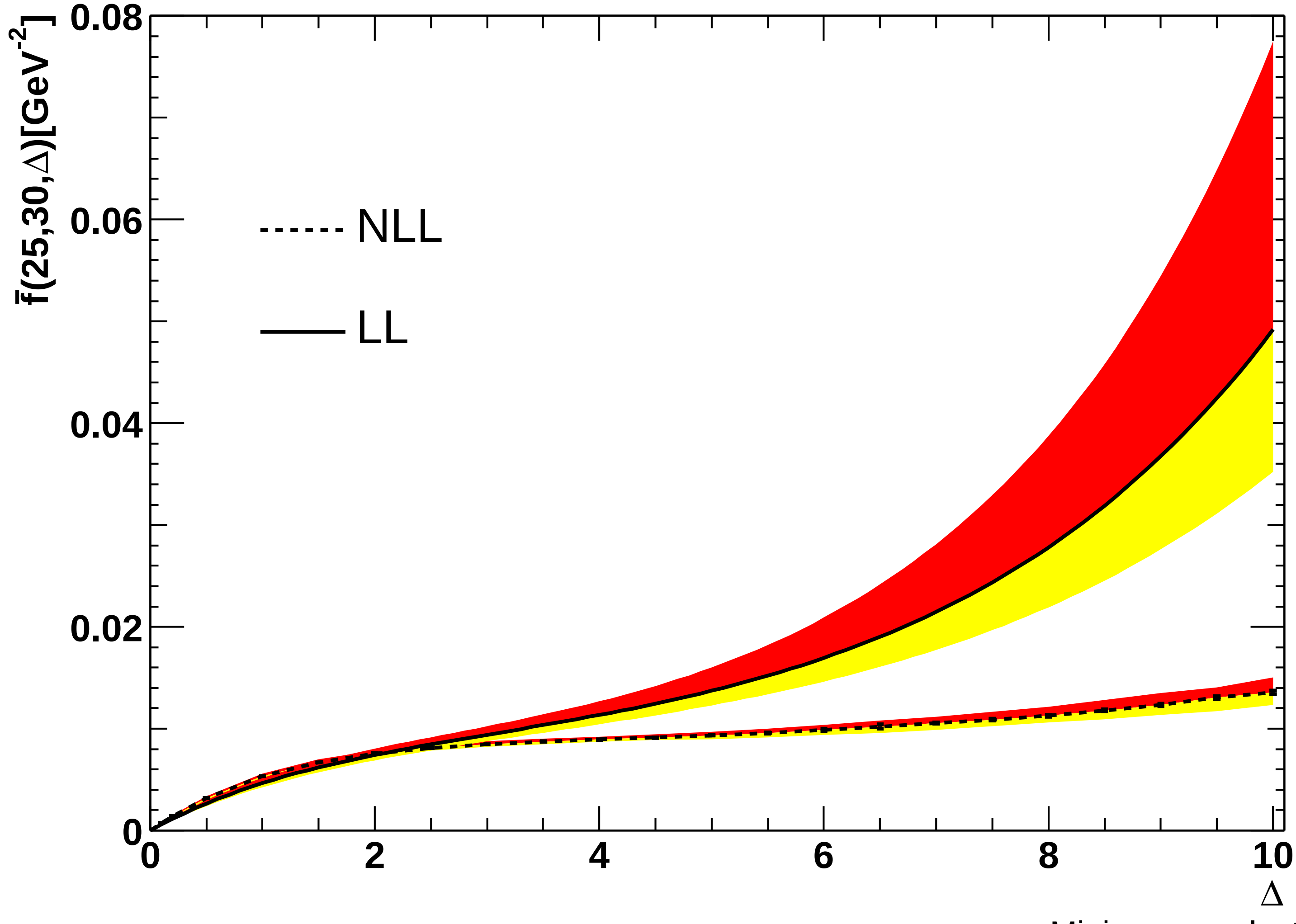
$\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM preserves conformal invariance at NLL



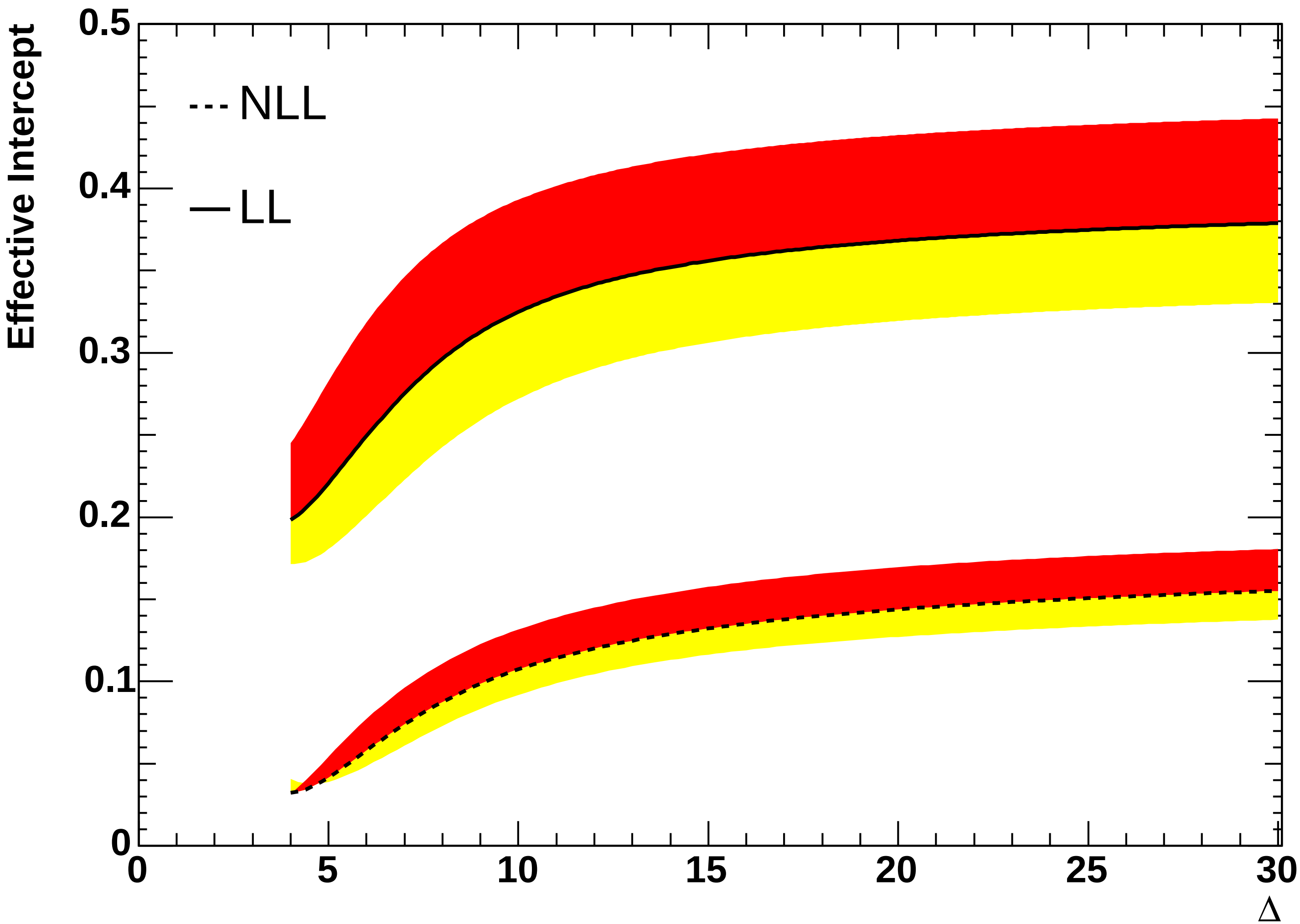
Dependence of f on Δ

QCD, $n_f = 4$, one loop running



BFKL Intercept

QCD, $n_f = 4$, one loop running



Conclusions

- We have solved the BFKL equation at full **Next–to-leading logarithmic accuracy** (No approximation: keeping all scale invariant and scale dependent terms, and full angular information.)
- . . . in a form that is directly suitable for calculation of cross sections (inclusion of impact factors)
- Explore **non-problem** of NLL BFKL
  - If you want to use an analytic approximation — use the right one!
- Method also applicable to the non-forward NLL BFKL equation - hope to report results soon.

Extra Slides

Iteration at NLL

Start from the BFKL equation

$$\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right)=\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right)+\int d^{2+2\epsilon}\mathbf{k}'\,\mathcal{K}\left(\mathbf{k}_a,\mathbf{k}'\right)f_\omega\left(\mathbf{k}',\mathbf{k}_b\right)$$

$$\mathcal{K}\left(\mathbf{k}_a,\mathbf{k}\right)=2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}\right)+\mathcal{K}_r\left(\mathbf{k}_a,\mathbf{k}\right)$$

Need all terms (IR) finite to be able to iterate: split the kernel $\mathcal{K}_r$ into two parts: a ϵ–dependent, $\mathcal{K}_r^{(\epsilon)}$, and a ϵ–independent, $\widetilde{\mathcal{K}}_r$

$$\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right)=\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right)+\int d^{2+2\epsilon}\mathbf{k}\,2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}\right)f_\omega\left(\mathbf{k},\mathbf{k}_b\right)$$
$$+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)+\int d^{2+2\epsilon}\mathbf{k}\,\widetilde{\mathcal{K}}_r\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right).$$

Iteration at NLL, 2

Introduce a slice in the phase space (no approximation)

$$
\begin{aligned}
\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) &= \delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right)+\int d^{2+2\epsilon}\mathbf{k}\,2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}\right)f_\omega\left(\mathbf{k},\mathbf{k}_b\right)\\
&+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\left(\theta\left(\mathbf{k}^2-\lambda^2\right)+\theta\left(\lambda^2-\mathbf{k}^2\right)\right)f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)\\
&+\int d^{2+2\epsilon}\mathbf{k}\,\widetilde{\mathcal{K}}_r\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)
\end{aligned}
$$

approximate $f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)\simeq f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right)$ for $|\mathbf{k}|<\lambda$

$$
\begin{aligned}
\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) &= \delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right)\\
&+\left\{2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\theta\left(\lambda^2-\mathbf{k}^2\right)\right\}f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right)\\
&+\int d^{2+2\epsilon}\mathbf{k}\,\left\{\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\theta\left(\mathbf{k}^2-\lambda^2\right)+\widetilde{\mathcal{K}}_r\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\right\}f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right).
\end{aligned}
$$

($\lambda\to 0$ limit can be obtained)

Iteration at NLL, 3

$$
\begin{aligned}
\left(\omega - \omega_0\left(\mathbf{k}_a^2, \lambda^2\right)\right) f_\omega\left(\mathbf{k}_a, \mathbf{k}_b\right) &= \delta^{(2)}\left(\mathbf{k}_a - \mathbf{k}_b\right) \\
&+ \int d^2\mathbf{k} \left(\frac{1}{\pi \mathbf{k}^2} \xi\left(\mathbf{k}^2\right) \theta\left(\mathbf{k}^2 - \lambda^2\right) + \widetilde{\mathcal{K}}_r\left(\mathbf{k}_a, \mathbf{k}_a + \mathbf{k}\right) \right) f_\omega\left(\mathbf{k}_a + \mathbf{k}, \mathbf{k}_b\right)
\end{aligned}
$$

$$
\begin{aligned}
\omega_0\left(\mathbf{q}^2, \lambda^2\right) &\equiv -\xi\left(|\mathbf{q}|\,\lambda\right) \ln \frac{\mathbf{q}^2}{\lambda^2} + \eta \\
\xi\left(\mathrm{X}\right) &\equiv \bar{\alpha}_s + \frac{\bar{\alpha}_s^2}{4} \left[\frac{4}{3} - \frac{\pi^2}{3} + \frac{5}{3} \frac{\beta_0}{N_c} - \frac{\beta_0}{N_c} \ln \frac{\mathrm{X}}{\mu^2} \right] \\
\eta &\equiv \bar{\alpha}_s^2 \frac{3}{2} \zeta(3).
\end{aligned}
$$

$$
\widetilde{\mathcal{K}}_r\left(\mathbf{q}, \mathbf{q}'\right) = \frac{\bar{\alpha}_s^2}{4\pi} \left\{ \text{6 lines of equations} \ldots \right\}.
$$

Iteration at NLL, 4

Iterate and take the inverse Mellin transform to find

$$
\begin{aligned}
f(\mathbf{k}_a, \mathbf{k}_b, \Delta) \quad &= \quad \exp\left(\omega_0\left(\mathbf{k}_a^2, \lambda^2, \mu\right)\Delta\right)\delta^{(2)}(\mathbf{k}_a - \mathbf{k}_b) \\
&+ \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int d^2\mathbf{k}_i \left[\frac{\theta\left(\mathbf{k}_i^2 - \lambda^2\right)}{\pi \mathbf{k}_i^2}\xi\left(\mathbf{k}_i^2, \mu\right) + \widetilde{\mathcal{K}}_r\left(\mathbf{k}_a + \sum_{l=0}^{i-1}\mathbf{k}_l, \mathbf{k}_a + \sum_{l=1}^{i}\mathbf{k}_l, \mu\right)\right] \\
&\times \int_0^{y_{i-1}} dy_i \ \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{i-1}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_{i-1} - y_i)\right] \\
&\times \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{n}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_n - 0)\right]\delta^{(2)}\left(\sum_{l=1}^{n}\mathbf{k}_l + \mathbf{k}_a - \mathbf{k}_b\right)
\end{aligned}
$$

JRA and A. Sabio Vera

Convergence

$$\bar{f}\left(k_a, k_b, \Delta\right) \;=\; \int_0^{2\pi} d\theta\, f\left(k_a, k_b, \theta, \Delta\right),$$

$k_a = 25$ GeV, $k_b = 30$ GeV, $\lambda = 1$ GeV