The HE limit of QCD as described by the NLL BFKL equation

Solving the BFKL Equation(s) by Iteration

Jeppe R. Andersen

HEP Group, Cavendish Lab

UNIVERSITY OF CAMBRIDGE

Outline of the talk

- The **High Energy Limit** of scattering processes and the **BFKL Resummation Framework**
- The solution of the **BFKL equation** at **Leading Logarithmic Accuracy**:
  - Eigen-values & E-functions (intrinsic features)
  - Iterative method
    What the iterative method gives extra
  - Example processes
- Formalism at **Next to Leading Logarithmic Accuracy**
  - One Perceived Problem
  - ...and the iterative solution
    Why one *has* to use a different approach at NLL
- **Non-forward** BFKL Equation at LLA and NLLA for **Diffractive processes**

QCD

Well understood for scattering processes with one hard scale. Total cross sections well described by:

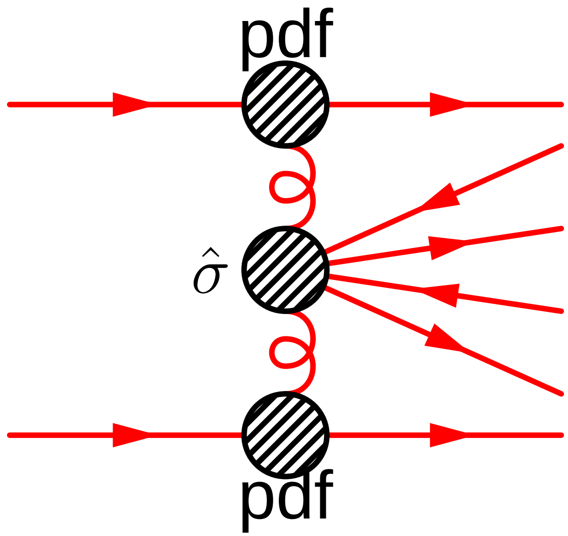
- Fixed (NL) order perturbative calculations
- DGLAP evolution of PDFs

Multi-Jet and Multi-Scale QCD: attempt to extend our understanding of perturbative QCD away from the study of total cross sections with one, super perturbative scale.

Excitement/Drawback: *not* 5-10% effects!

Surprises may lurk round the corner

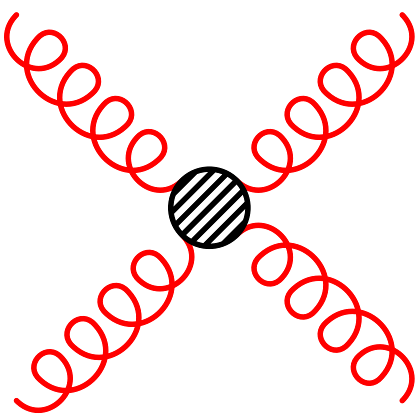
High Energy Limit



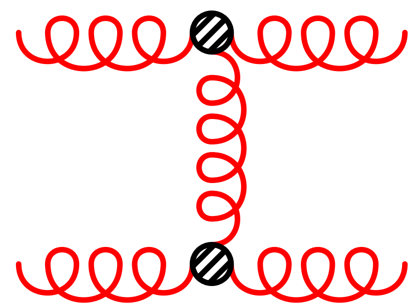
High energy limit:

$$\frac{\hat{s}}{|\hat{t}|} \longrightarrow \infty$$

$|\mathcal{M}|^2$ factorises.

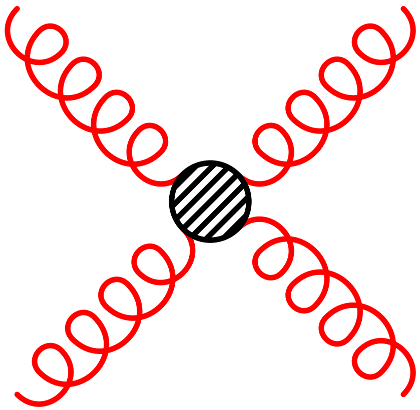


$\xrightarrow{\text{High Energy Limit}}$



Diagrams with a t-channel gluon exchange dominate the cross section.

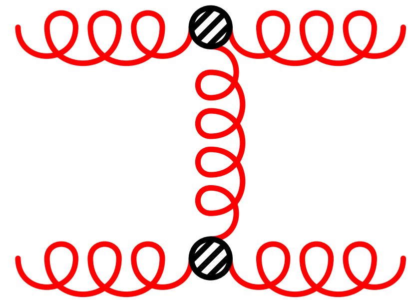
HE limit and Dijet Production



High Energy Limit

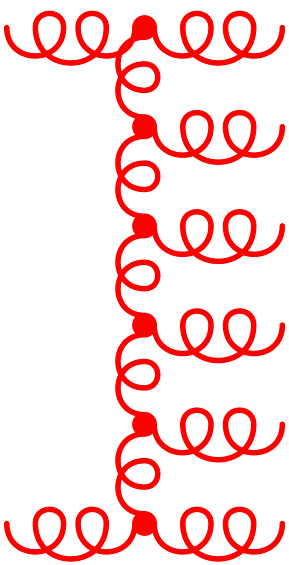
$$\xrightarrow{\quad\quad}$$

$$\hat{s}/|\hat{t}| \to \infty$$



Large logarithm in multi-gluon production

$P_{Ta}, \Delta y$



$P_{Tb}, 0$

$$p_{Ti}^2 \sim p_{Ta}^2 \sim p_{Tb}^2$$

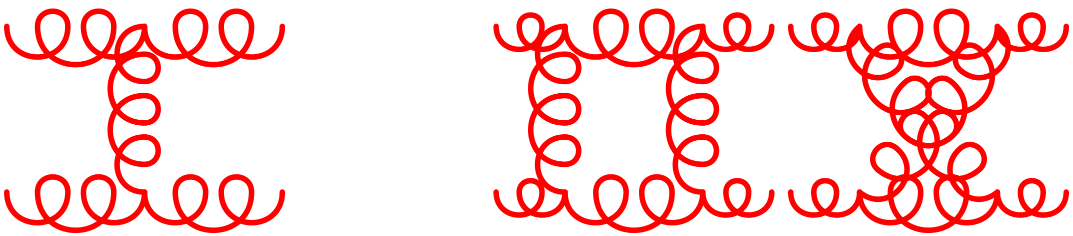
$$\hat{s}_{ij} \gg p_{Ti}^2$$

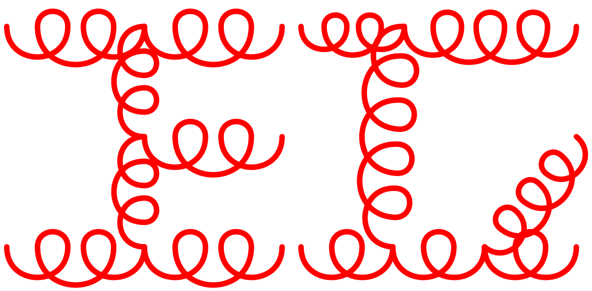
BFKL resums to all orders terms in the perturbative expansion of the form

$$\left(\alpha_s \ln \frac{\hat{s}_{ij}}{|\hat{t}_i|}\right) \sim (\alpha_s \Delta y)$$

BFKL at LLA

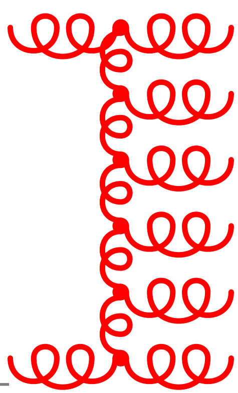
Which diagrams contribute?





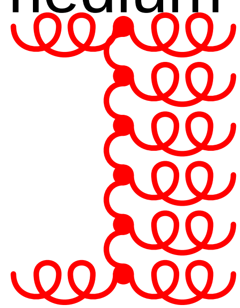
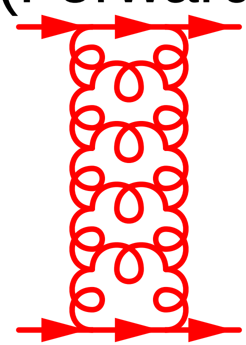
etc.

All these contributions can be calculated using effective vertices and propagators for the **reggeized gluon**.



BFKL formalism

- **BFKL** (**B**alitskii, **F**adin, **K**uraev, **L**ipatov): resummation of **large logarithms** in the perturbation series for QCD processes with **two large** (perturbative) and **disparate** energy scales $\hat{s} \gg |\hat{t}|$ ($\hat{s}$: E^2, $|\hat{t}|$: $p_\perp^2$)

Structure Functions	Forward Physics @ Hadron Colliders (Colour Octet Exchange)	Diffraction (Colour Singlet Exchange)
Small x	Large Rapidity (Forward) medium x 	Large Rapidity (Forward) 

BFKL formalism

- The cross section for the process $A + B \to A' + B'$ factorises as

$$\sigma(s) = \int \frac{d^2\mathbf{k}_a}{2\pi\mathbf{k}_a^2} \int \frac{d^2\mathbf{k}_b}{2\pi\mathbf{k}_b^2}\, \Phi_A(\mathbf{k}_a)\, f\left(\mathbf{k}_a, \mathbf{k}_b, \Delta = \ln\frac{s}{s_0}\right) \Phi_B(\mathbf{k}_b)$$

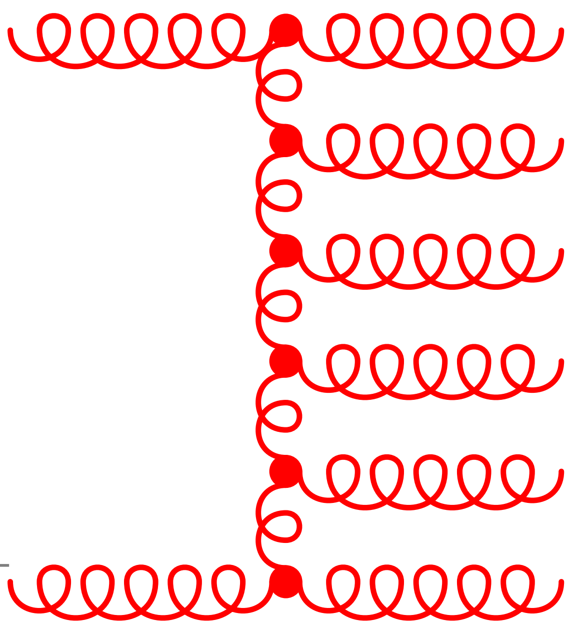
- $\Phi_A(\mathbf{k}_a), \Phi_B(\mathbf{k}_b)$ process dependent *impact factors* (calculated for many process at LL and for e.g. gg and (ongoing) $\gamma^*\gamma^*$ scattering at NLL)
- $f\,(\mathbf{k}_a, \mathbf{k}_b, \Delta)$ process independent *Gluon Green's function*

BFKL at LLA

gluon-gluon scattering:

$$\frac{\mathrm{d}\hat{\sigma}_{gg}}{\mathrm{d}^2\vec{p}_{T_a}\mathrm{d}^2\vec{p}_{T_b}} = \underbrace{\left[\frac{C_A\alpha_s}{p_{T_a}^2}\right]}_{\text{QCD IF}} \underbrace{f(\vec{q}_a, \vec{q}_b, \Delta y)}_{\text{BFKL}} \underbrace{\left[\frac{C_A\alpha_s}{p_{T_b}^2}\right]}_{\text{QCD IF}}$$

$$\vec{q}_a = \vec{p}_{T_a}, \vec{q}_b = -\vec{p}_{T_b}$$



Resum leading logarithms contributing to $f(\vec{q}_a, \vec{q}_b, \Delta y)$.

The BFKL Equation

The Gluon Green's function fulfil (to LLA and NLLA) the BFKL equation (in dim. regularisation $(D = 4 + 2\epsilon)$):

$$\omega f_\omega \left(\mathbf{k}_a, \mathbf{k}_b\right) = \delta^{(2+2\epsilon)} \left(\mathbf{k}_a - \mathbf{k}_b\right) + \int d^{2+2\epsilon}\mathbf{k}' \mathcal{K}_\epsilon \left(\mathbf{k}_a, \mathbf{k}'\right) f_\omega \left(\mathbf{k}', \mathbf{k}_b\right)$$

where the BFKL kernel $\mathcal{K}\left(\mathbf{k}_a, \mathbf{k}'\right)$ is calculated to LLA or NLLA respectively. At LL the kernel is **conformal invariant** (no running coupling) with **eigenfunctions** $\mathbf{k}^{2(\gamma-1)}$. Use (transverse) Mellin transform!

$$\int d^2\mathbf{k}' \, \mathcal{K}\left(\mathbf{k}, \mathbf{k}'\right) \, \mathbf{k}^{2(\gamma-1)} = \frac{N_c \alpha_s}{\pi} \chi^{\mathrm{LL}}(\gamma) \mathbf{k}^{2(\gamma-1)}$$

$$\omega(\gamma) = \langle\gamma|\mathcal{K}(k,k)|\gamma\rangle$$

$$f_\omega(\mathbf{k}_a, \mathbf{k}_b) = \sum_\gamma \frac{\langle\gamma, \mathbf{k}_b|\gamma, \mathbf{k}_a\rangle}{\omega - \omega(\gamma)}$$

The BFKL Equation at LLA

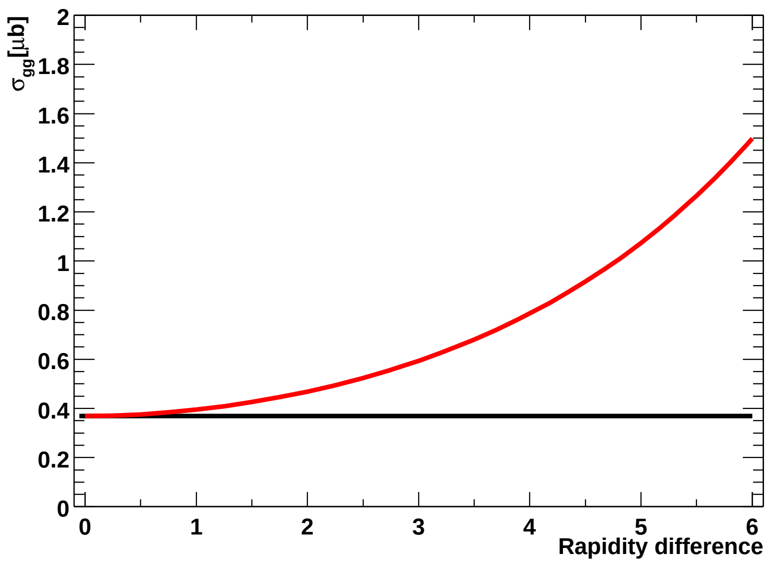
Analytic solution for angular averaged gluon Green's function

$$\bar{f}\left(k_a, k_b, \Delta\right) = \frac{1}{k_a k_b} \int_0^\infty d\nu \left(\frac{k_a^2}{k_b^2}\right)^{i\nu} e^{\bar{\alpha}_s \Delta \chi_0(\nu)}$$

with the LL eigenvalue

$$\chi_0(\nu) = -2\,\mathrm{Re}\left\{\psi\left(\frac{1}{2} + i\nu\right) - \psi(1)\right\}.$$

Cross section vs. rapidity difference



BFKL rise in cross section! Integrated over the **full k phase space** for gluon emission and allowing **any number** of gluons to radiate!!!

$$\hat{\sigma}_{gg} \to \frac{\pi C_A^2 \alpha_s^2}{2 P_{T,\min}^2} \frac{e^{\lambda \Delta y}}{\sqrt{\pi B \Delta y}}, B = 14\zeta(3)\bar{\alpha}_s, \quad \lambda = \frac{\alpha_s C_A}{\pi} 4 \ln 2 \approx 0.45$$

The BFKL Equation at NLLA

- Both the *trajectory* $\omega(-\mathbf{k}_a^2)$ and the *real emission kernel* $\mathcal{K}_r$ are significantly more complicated than at LL
- Takes into account fermions and running coupling effects
- Furthermore, the impact factors at NLL are similarly complicated, and a fully analytic approach for cross sections seems almost hopeless
- We will propose a generalisation of the LL iterative solution that will solve the BFKL equation at NLL accuracy.
- But first the story so far…

The NLL BFKL Story So Far

- BFKL equation at LL put forward and solved in 1978.
  - **non-forward** equation solved five years later by L. Lipatov
- 8-10 years effort to calculate the BFKL kernel at NLLA ended in 1998
  - Initial results were discouraging. NLL kernel applied to LL eigenfunctions lead to huge and unstable corrections.
  - We will see why this analysis is invalid.
- Calculation of the **non-forward** kernel finished Dec. 2004 by V. Fadin and collaborators.

BFKL at NLLA

- Two new effects appear:
  - Fermions
  - Running Coupling

Conformal invariance **broken** (in QCD for $\beta \neq 0$) — Eigenfunctions **unknown**. Analyse what happens **if we pretend** the LL eigenfunctions are also eigenfunctions at NLL.

$$\omega^{\mathrm{NLL}}(\gamma) = \langle \gamma, \mathbf{k}_b | \mathcal{K}^{\mathrm{NLL}}(\mathbf{k}_b, \mathbf{k}_a) | \gamma, \mathbf{k}_a \rangle$$

$$f_\omega(\mathbf{k}_a, \mathbf{k}_b) = \sum_\gamma \frac{\langle \gamma, \mathbf{k}_b | \gamma, \mathbf{k}_a \rangle}{\omega - \omega^{\mathrm{NLL}}(\gamma)}$$

Leading Log tools at NLL

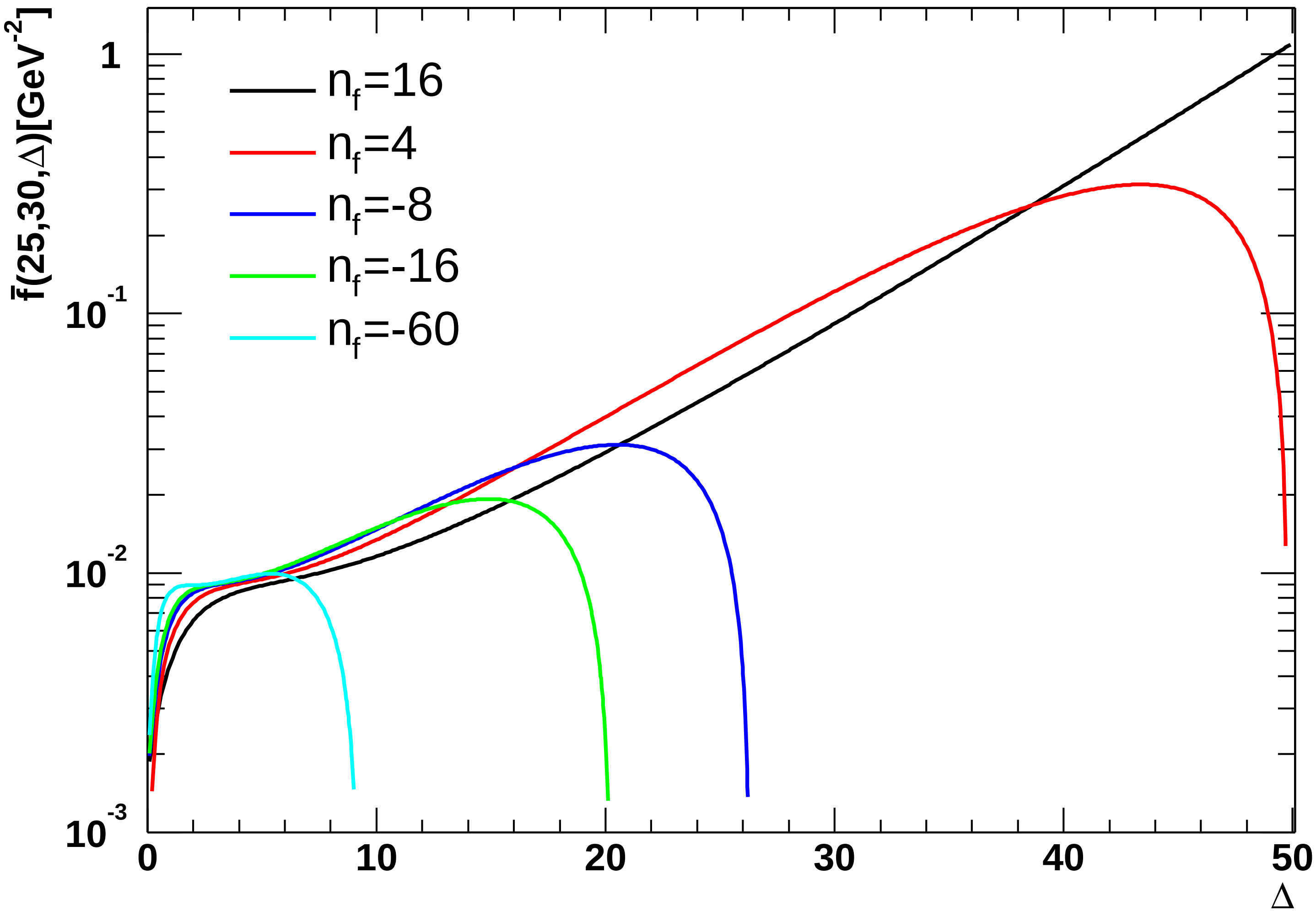
$$\omega^{\rm NLL}(\gamma) = \int {\rm d}^{D-2}\mathbf{k}\, \mathcal{K}^{\rm NLL}(\mathbf{k}_a,\mathbf{k}) \left(\frac{\mathbf{k}^2}{\mathbf{k}_a^2}\right)^{\gamma-1}$$
$$= \frac{\alpha_s(\mathbf{k}_a^2)N}{\pi}\left(\chi^{\rm LL}(\gamma) + \chi^{\rm NLL}(\gamma)\frac{\alpha_s(\mathbf{k}_a^2)N}{\pi}\right)$$

$$\chi^{\rm NLL}(\gamma) = -\frac{1}{4}\Big[\left(\frac{11}{3}-\frac{2}{3}\frac{n_f}{N}\right)\frac{1}{2}\left(\chi^{\rm LL}(\gamma)-\psi'(\gamma)+\psi'(1-\gamma)\right)$$
$$-6\zeta(3)+\frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left(3+\left(1+\frac{n_f}{N^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)$$
$$-\left(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10}{9}\frac{n_f}{N}\right)\chi^{\rm LL}(\gamma)-\psi''(\gamma)-\psi''(1-\gamma)-\frac{\pi^3}{\sin(\pi\gamma)}+4\phi(\gamma)\Big],$$

let us pretend:

$$\bar{f}(k_a,k_b,\Delta) = \frac{1}{k_b^2}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{{\rm d}\gamma}{2\pi i}\, e^{\Delta\omega^{\rm NLL}(\gamma)}\left(\frac{k_b^2}{k_a^2}\right)^{\gamma}$$

Leading Log tools at NLL

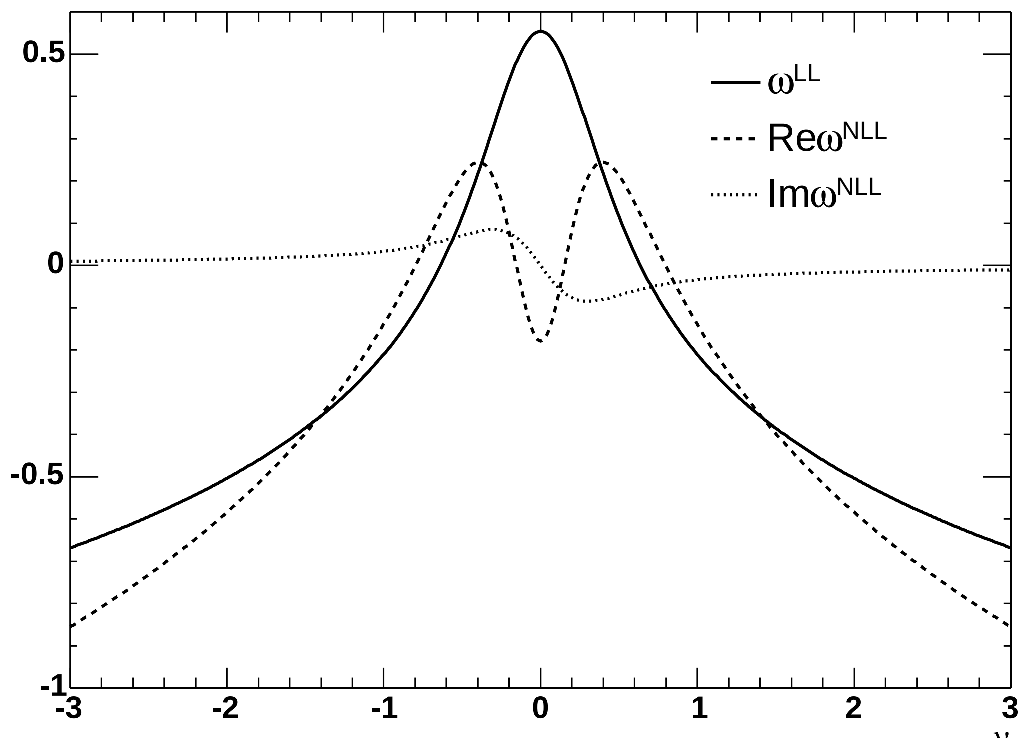


!!this would be a major catastrophe!!

Leading Log tools at NLL

$$\bar{f}(k_a, k_b, \Delta) = \frac{1}{k_b^2} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\mathrm{d}\gamma}{2\pi i} \, e^{\Delta\omega^{\mathrm{NLL}}(\gamma)} \left(\frac{k_b^2}{k_a^2}\right)^{\gamma}$$

$$\gamma = \frac{1}{2} + i\nu$$



Read the small print

Fadin and Lipatov say

Almost all the terms in the right hand side of eq. (12) except the contribution

$$\Delta(\gamma) = \frac{\alpha_s^2(\mu^2) N_c^2}{4\pi^2} \left(\frac{11}{3} - \frac{2n_f}{3N_c} \right) \frac{1}{2} \left(\psi'(\gamma) - \psi'(1-\gamma) \right)$$

are symmetric to the transformation $\gamma \leftrightarrow 1-\gamma$. Moreover, it is possible to cancel $\Delta(\gamma)$ if one would redefine the function $q^{2(\gamma-1)}$ by including in it the logarithmic factor $\left(\frac{\alpha_s(q^2)}{\alpha_s(\mu^2)} \right)^{-1/2}$.

This would **remove the imaginary part**, and therefore also **remove the oscillations**.

What to believe?

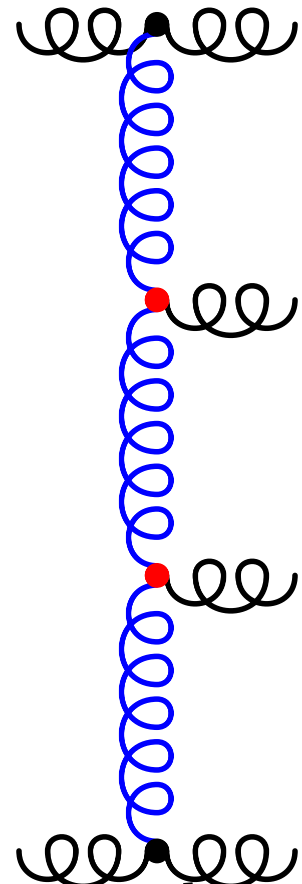
Iterative Solution at NLLA

We propose an iterative approach to the BFKL equation at NLLA that solves the equation with *no approximations*

- Directly in the physical rapidity and transverse momentum space
  (avoids the use of the troublesome Mellin transform completely)
- The right language for use of impact factors (physics predictions!)
- Hopeful in extending the approach to final state studies like at LL
- Expresses the solution in terms of effective vertices and no-emission probabilities (physical insight into the BFKL solution at NLLA!)

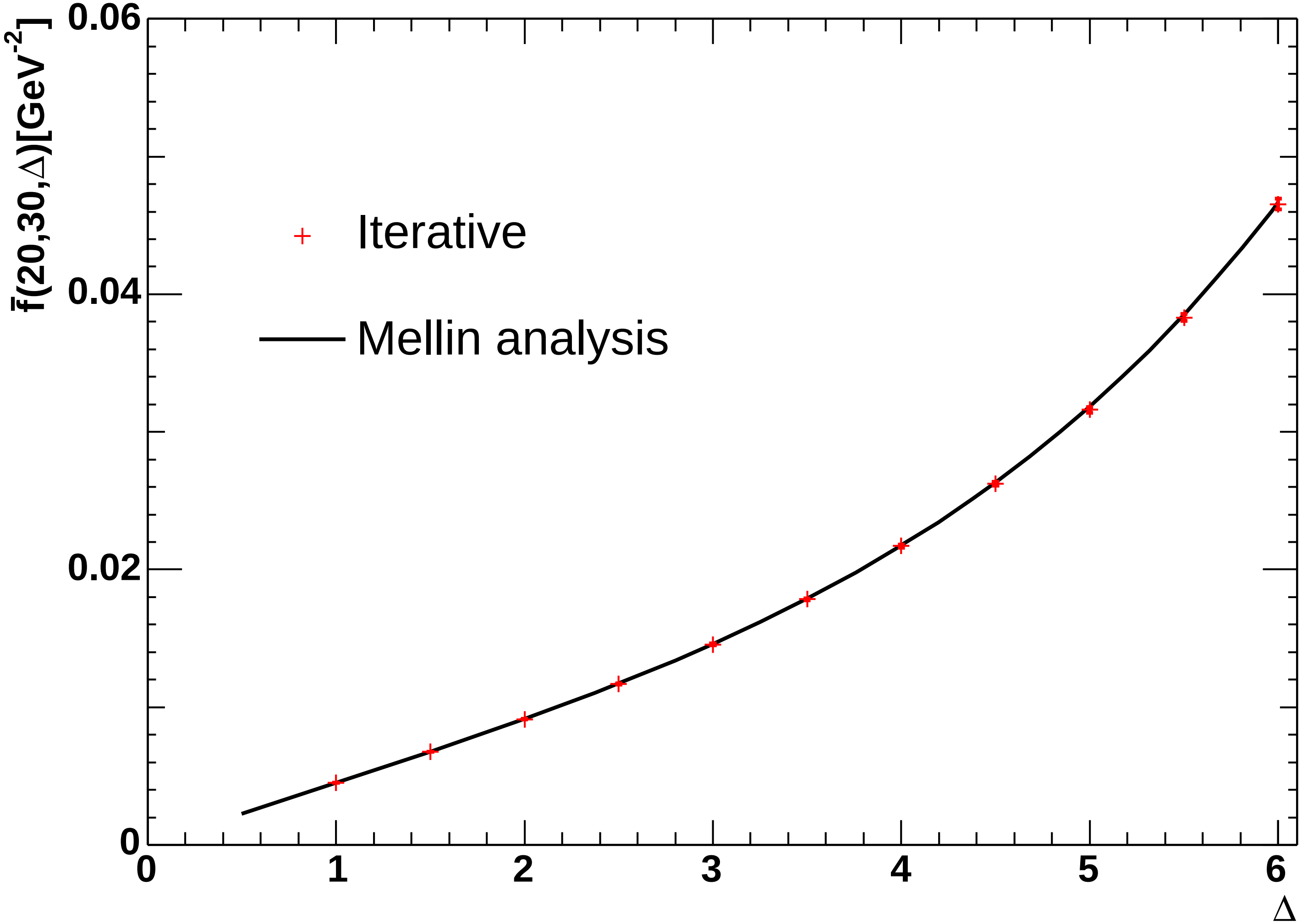
Enter Iteration at NLLA

$$
\begin{aligned}
f(\mathbf{k}_a, \mathbf{k}_b, \Delta) \; &= \; \exp\left(\omega_0\left(\mathbf{k}_a^2, \lambda^2, \mu\right)\Delta\right)\delta^{(2)}(\mathbf{k}_a - \mathbf{k}_b) \\
&+ \sum_{n=1}^{\infty}\prod_{i=1}^{n}\int d^2\mathbf{k}_i \int_0^{y_{i-1}} dy_i \left[V\left(\mathbf{k}_i, \mathbf{k}_a + \sum_{l=0}^{i-1}\mathbf{k}_l, \mu\right)\right] \\
&\times \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{i-1}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_{i-1} - y_i)\right] \\
&\quad \times \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{n}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_n - 0)\right] \\
&\quad \times \delta^{(2)}\left(\sum_{l=1}^{n}\mathbf{k}_l + \mathbf{k}_a - \mathbf{k}_b\right)
\end{aligned}
$$



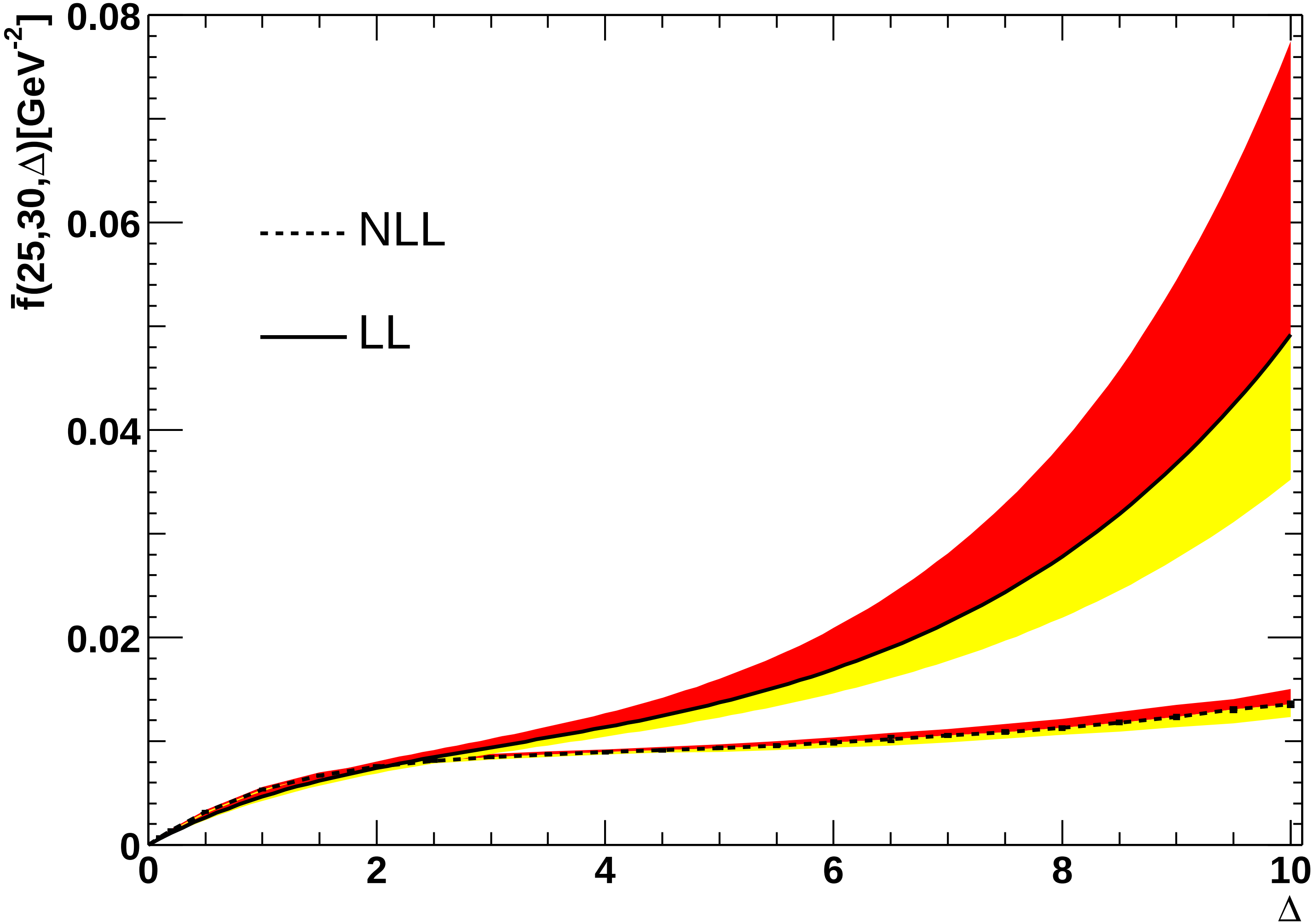
$\mathcal{N}=4$ SYM

$\mathcal{N}=4$ SYM preserves conformal invariance at NLL



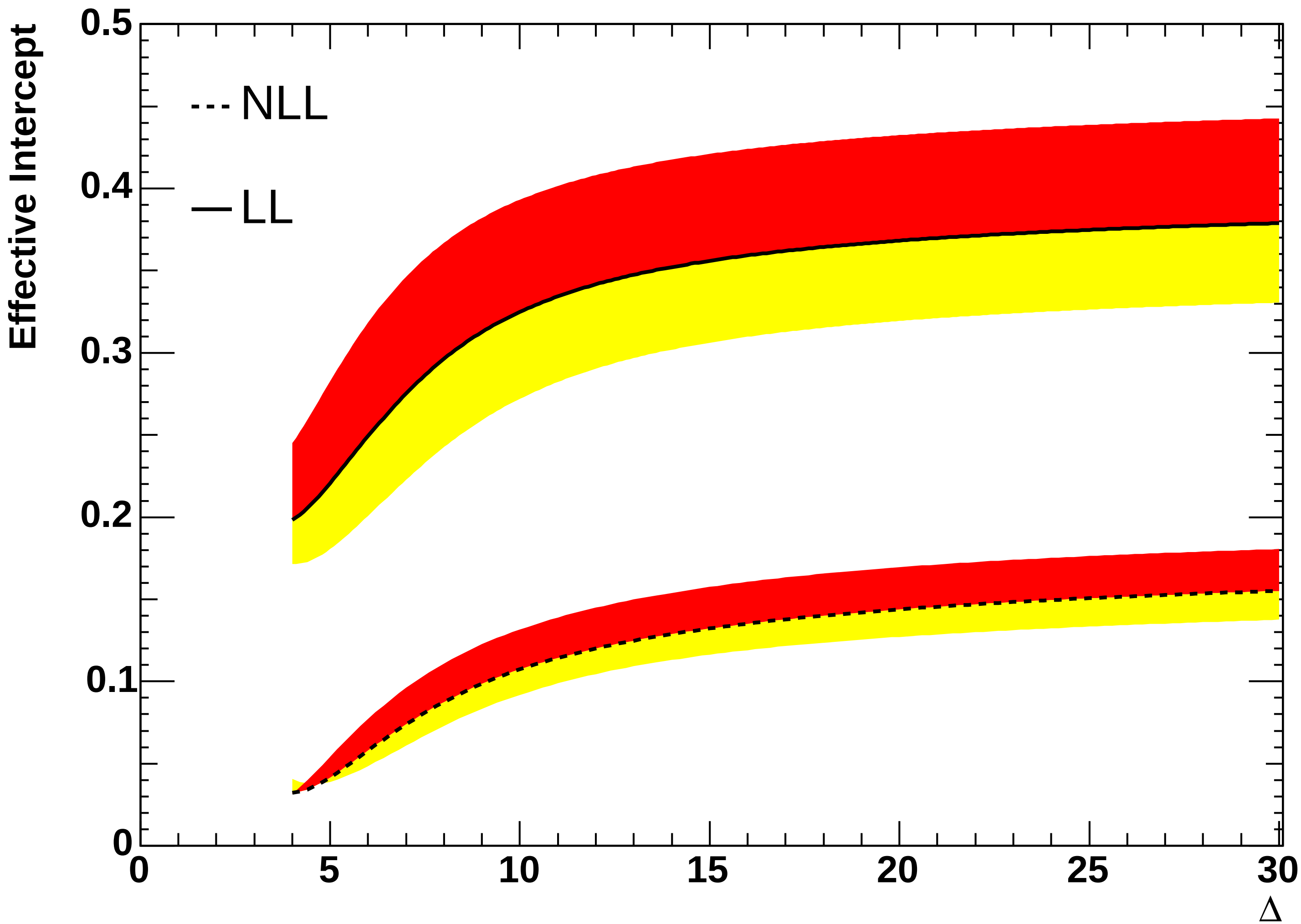
Dependence of f on Δ

QCD, $n_f = 4$, one loop running

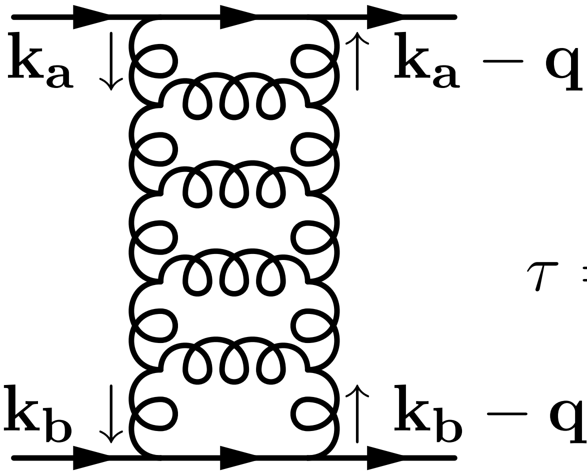


BFKL Intercept

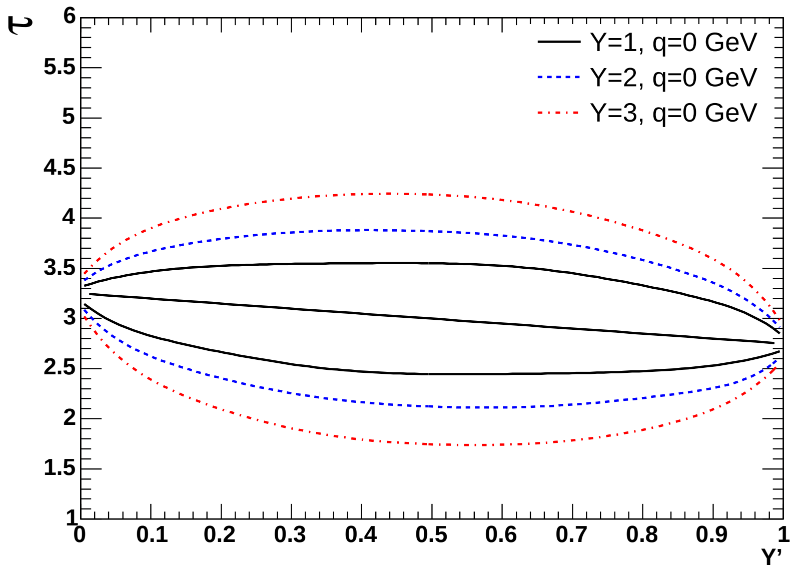
QCD, $n_f = 4$, one loop running

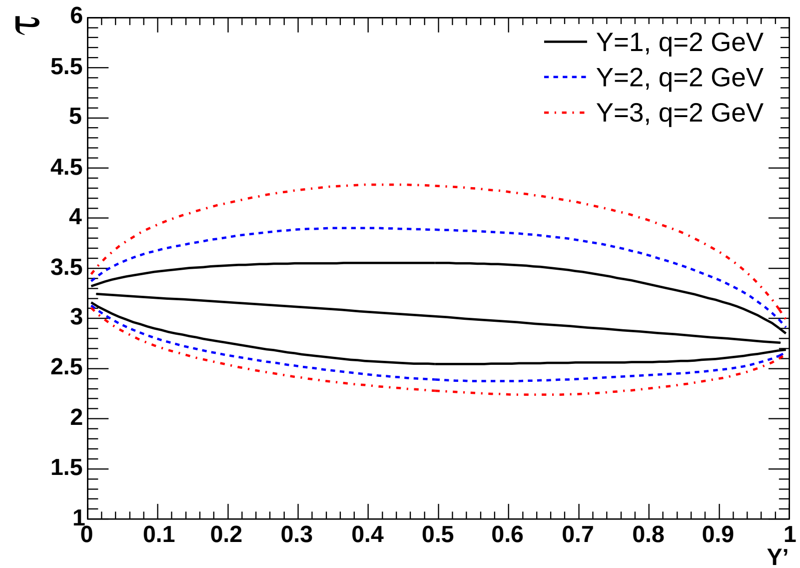


Diffusion and Diffraction at LLA



$$\tau = \ln\left(\left(k_a + \sum k_i\right)^2 / \mathrm{GeV}^2\right)$$



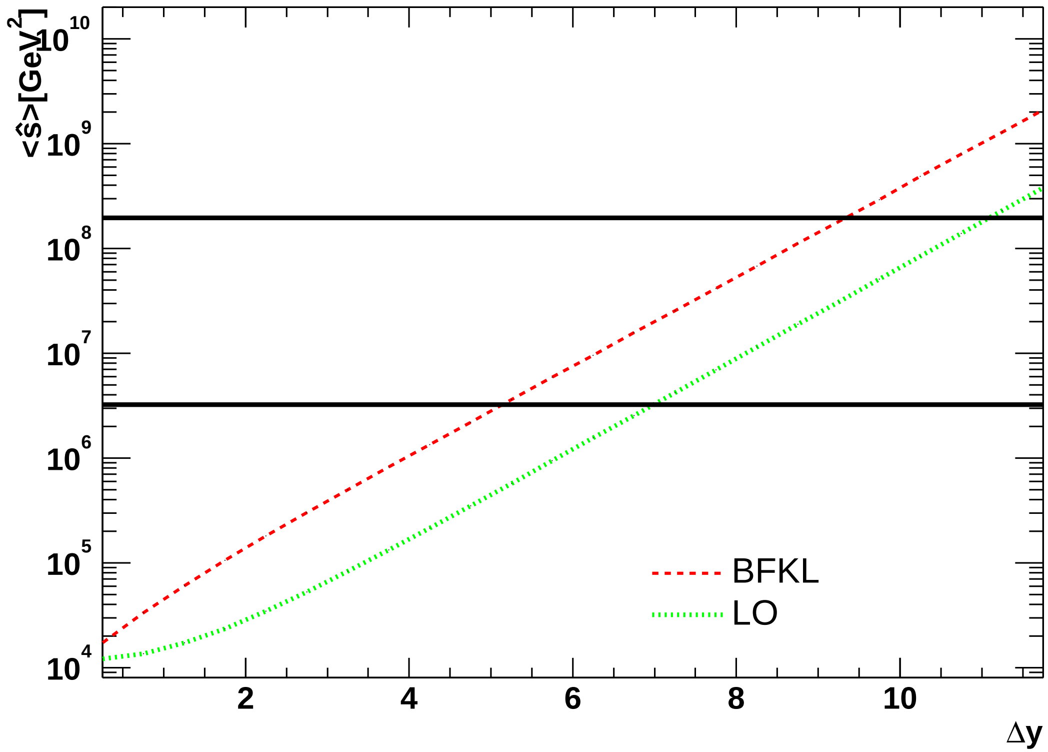


Conclusions

- We have solved the BFKL equation at full **Next–to-leading logarithmic accuracy** (No approximation: keeping all scale invariant and scale dependent terms, and full angular information.)
- . . . in a form that is directly suitable for calculation of cross sections (inclusion of impact factors)
- Explore **non-problem** of NLL BFKL
  - If you want to use an analytic approximation — use the right one!
- Will extend study to **final states at NLL** - necessary for phenomenology at full NLL accuracy (resum **only available phase space**)
- Method also applicable to the non-forward (NLL) BFKL equation

Extra Slides

Energy Consumption of BFKL evolution



$$\hat{s}_{\mathrm{BFKL}} \propto \hat{s}_{\mathrm{LO}} \frac{1}{(1-e^{-\delta y})^2}$$

Large effects - Resum only the phase space accessible at a given collider!

Iteration at NLL

Start from the BFKL equation

$$\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) = \delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right) + \int d^{2+2\epsilon}\mathbf{k}'\,\mathcal{K}\left(\mathbf{k}_a,\mathbf{k}'\right) f_\omega\left(\mathbf{k}',\mathbf{k}_b\right)$$

$$\mathcal{K}\left(\mathbf{k}_a,\mathbf{k}\right) = 2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\,\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}\right) + \mathcal{K}_r\left(\mathbf{k}_a,\mathbf{k}\right)$$

Need all terms (IR) finite to be able to iterate: split the kernel $\mathcal{K}_r$ into two parts: a ϵ–dependent, $\mathcal{K}_r^{(\epsilon)}$, and a ϵ–independent, $\widetilde{\mathcal{K}}_r$

$$\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) = \delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right) + \int d^{2+2\epsilon}\mathbf{k}\,2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}\right) f_\omega\left(\mathbf{k},\mathbf{k}_b\right)$$

$$+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right) f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right) + \int d^{2+2\epsilon}\mathbf{k}\,\widetilde{\mathcal{K}}_r\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right) f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right).$$

Iteration at NLL, 2

Introduce a slice in the phase space (no approximation)

$$\begin{aligned}\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) &= \delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right)+\int d^{2+2\epsilon}\mathbf{k}\,2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}\right)f_\omega\left(\mathbf{k},\mathbf{k}_b\right)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\left(\theta\left(\mathbf{k}^2-\lambda^2\right)+\theta\left(\lambda^2-\mathbf{k}^2\right)\right)f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\widetilde{\mathcal{K}}_r\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)\end{aligned}$$

approximate $f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)\simeq f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right)$ for $|\mathbf{k}|<\lambda$

$$\begin{aligned}\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) &= \delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right)\\ &+\left\{2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\theta\left(\lambda^2-\mathbf{k}^2\right)\right\}f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\left\{\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\theta\left(\mathbf{k}^2-\lambda^2\right)+\widetilde{\mathcal{K}}_r\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\right\}f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right).\end{aligned}$$

($\lambda\to 0$ limit can be obtained)

Iteration at NLL, 3

$$
\begin{aligned}
\left(\omega - \omega_0\left(\mathbf{k}_a^2, \lambda^2\right)\right) f_\omega\left(\mathbf{k}_a, \mathbf{k}_b\right) &= \delta^{(2)}\left(\mathbf{k}_a - \mathbf{k}_b\right) \\
&+ \int d^2\mathbf{k} \left(\frac{1}{\pi \mathbf{k}^2} \xi\left(\mathbf{k}^2\right) \theta\left(\mathbf{k}^2 - \lambda^2\right) + \widetilde{\mathcal{K}}_r\left(\mathbf{k}_a, \mathbf{k}_a + \mathbf{k}\right) \right) f_\omega\left(\mathbf{k}_a + \mathbf{k}, \mathbf{k}_b\right)
\end{aligned}
$$

$$
\begin{aligned}
\omega_0\left(\mathbf{q}^2, \lambda^2\right) &\equiv -\xi\left(|\mathbf{q}|\,\lambda\right) \ln \frac{\mathbf{q}^2}{\lambda^2} + \eta \\
\xi\left(\mathrm{X}\right) &\equiv \bar{\alpha}_s + \frac{\bar{\alpha}_s^2}{4}\left[\frac{4}{3} - \frac{\pi^2}{3} + \frac{5}{3}\frac{\beta_0}{N_c} - \frac{\beta_0}{N_c} \ln \frac{\mathrm{X}}{\mu^2}\right] \\
\eta &\equiv \bar{\alpha}_s^2 \frac{3}{2} \zeta(3).
\end{aligned}
$$

$$
\widetilde{\mathcal{K}}_r\left(\mathbf{q}, \mathbf{q}'\right) = \frac{\bar{\alpha}_s^2}{4\pi}\left\{\text{6 lines of equations}\ldots\right\}.
$$

Iteration at NLL, 4

Iterate and take the inverse Mellin transform to find

$$
\begin{aligned}
f(\mathbf{k}_a, \mathbf{k}_b, \Delta) &= \exp\left(\omega_0\left(\mathbf{k}_a^2, \lambda^2, \mu\right)\Delta\right)\delta^{(2)}(\mathbf{k}_a - \mathbf{k}_b) \\
&+ \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int d^2\mathbf{k}_i \left[\frac{\theta\left(\mathbf{k}_i^2 - \lambda^2\right)}{\pi \mathbf{k}_i^2}\xi\left(\mathbf{k}_i^2, \mu\right) + \widetilde{\mathcal{K}}_r\left(\mathbf{k}_a + \sum_{l=0}^{i-1}\mathbf{k}_l, \mathbf{k}_a + \sum_{l=1}^{i}\mathbf{k}_l, \mu\right)\right] \\
&\times \int_0^{y_{i-1}} dy_i \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{i-1}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_{i-1} - y_i)\right] \\
&\times \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{n}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_n - 0)\right]\delta^{(2)}\left(\sum_{l=1}^{n}\mathbf{k}_l + \mathbf{k}_a - \mathbf{k}_b\right)
\end{aligned}
$$

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Convergence

$$\bar{f}\,(k_a, k_b, \Delta) \;\; = \;\; \int_0^{2\pi} d\theta\, f\,(k_a, k_b, \theta, \Delta)\,,$$

$k_a = 25$ GeV, $k_b = 30$ GeV, $\lambda = 1$ GeV