Solving Pure Yang-Mills Theory in 2+1 Dimensions

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Motivation (Why is $YM_{(2+1)}$ interesting?)

- Interesting in its own right

$YM_{(1+1)}$	$YM_{(2+1)}$	$YM_{(3+1)}$
No propagating degrees of freedom; exactly solvable ('t Hooft '74)	Propagating degrees of freedom, nontrivial. Exactly solvable? (Polyakov '80)	Highly nontrivial; difficult

- A real physical context for $YM_{(2+1)}$
  - Mass gap of $YM_{(2+1)}$ ≈ magnetic screening mass of $YM_{(3+1)}$ at high temperature
  - Possible applications in condensed matter physics (high-T_c superconductivity)

Preliminaries and Summary

- We work in the Hamiltonian formalism
- The new ingredient is provided by the computation of the new nontrivial form of the ground state wave-functional. This wave-functional correctly interpolates between asymptotically free regime and low energy confining physics
- With this vacuum state it is possible to (quantitatively) demonstrate important observable features of the theory:
  - Signals of confinement: area law, string tension, mass gap
  - Compute the spectrum of glueball states
- Excellent agreement with available lattice data

$YM_{(2+1)}$ in the Hamiltonian Formalism

- We consider (2+1)D SU(N) pure YM theory with the Hamiltonian

$$\mathcal{H}_{YM} \equiv T + V = \int \mathrm{Tr}\left(g_{YM}^2 {E_i}^2 + \frac{1}{g_{YM}^2} B^2 \right)$$

- We choose the temporal gauge $A_0=0$
- E^a_i is the momentum conjugate to A^a_i ; i=1,2, a=1,2,…,N^2-1
  - Quantize: $E_i^a \rightarrow -i\frac{\delta}{\delta A_i^a}$
- Time-independent gauge transformations preserve $A_0=0$ gauge condition and gauge fields A_i transform as

$$A_i \rightarrow gAg^{-1} - \partial_i g g^{-1}, \qquad g \in SU(N)$$

- Gauss' law implies that observables and physical states are gauge invariant

$YM_{(2+1)}$ in the Hamiltonian Formalism (cont'd.)

- $YM_{(2+1)}$ is *superrenormalizable*
- Coupling constant is dimensionful: $[g^2_{YM}] = \text{mass}$
- It is convenient to introduce new massive parameter

$$m = \frac{g^2_{YM} N}{2\pi} \qquad \sim \; \text{'t Hooft coupling}$$

- Regularization is needed:
  - We use *Karabali, Kim and Nair formalism* (hep-th/9705087, hep-th/9804132, hep-th/0007188) which can be summarized as: local gauge-invariant variables + covariant point-splitting regularization

Vacuum Wave-Functional

- In general, for the vacuum wave-functional we may write

$$\Psi_0 = e^P$$

- In principle, *P* can be any functional which is gauge invariant, as well as invariant under space-time symmetries ($J^{PC} = 0^{++}$)
- We want to solve Schrödinger equation to *quadratic order* in magnetic field *B*, therefore we take the most general *gauge invariant* ansatz which contains *all* terms quadratic in B

$$\Psi_0 = \exp\left(-\frac{1}{2g_{YM}^2 m}\int tr\, B\, K\left(\frac{D^2}{4m^2}\right)B + \dots\right)$$

- The Gaussian part of the vacuum wave functional contains a (non-trivial) kernel *K* which will be determined by the solution of Schrödinger equation

Vacuum Wave-Functional (cont'd.)

- Asymptotic behavior of the vacuum state:
  - In the UV we expect to recover the standard perturbative result

$$\Psi_0^{UV} \mapsto \exp\left(-\frac{1}{2g_{YM}^2}\int B^a \frac{1}{|p|} B^a\right)$$

$$K \rightarrow \frac{2m}{p} \qquad \text{as} \qquad p \rightarrow \infty$$

  - In the IR we expect

$$\Psi_0^{IR} \mapsto \exp\left(-\frac{1}{2g_{YM}^2 m}\int \mathrm{Tr}\, B^2\right)$$

$$K \rightarrow 1 \qquad \text{as} \qquad p \rightarrow 0$$

Schrödinger Equation

- The Schrödinger equation takes the form

$$\mathcal{H}_{YM}\Psi_0 = E_0\Psi_0 = \left[E_0 + \int tr\, B(\mathcal{R})B + \ldots\right]\Psi_0$$

- By careful computation we find the differential equation for the kernel *K(L)*

$$\mathcal{R} = -K(L) - \frac{L}{2}\frac{d}{dL}[K(L)] + LK(L)^2 + 1 = 0$$

- This may be compared to *U(1)* theory without matter in which case we obtain an algebraic equation describing free photons

$$LK^2(L) + 1 = 0$$

$$K(L) = \pm\frac{1}{\sqrt{-L}} = \frac{2m}{p}$$

Vacuum Solution

- The differential equation for kernel is of Riccati type and, by a series of redefinitions, it can be recast as a Bessel equation.

$$K(L) = \frac{1}{\sqrt{L}} \frac{C J_2(4\sqrt{L}) + Y_2(4\sqrt{L})}{C J_1(4\sqrt{L}) + Y_1(4\sqrt{L})}$$

- *The only normalizable wave functional is obtained for* $C \to \infty$ *, which is also the only case that has both the correct UV behavior appropriate to asymptotic freedom as well as the correct IR behavior appropriate to confinement and mass gap!*

- This solution is of the form

$$K(L) = \frac{1}{\sqrt{L}} \frac{J_2(4\sqrt{L})}{J_1(4\sqrt{L})}$$

String tension and correlators

- We may now compute equal-time correlators as

$$\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle \;\sim\; \int [dA]\; |\Psi_0|^2\; \mathcal{O}(x)\mathcal{O}(y)$$

- Because of the Gaussian nature of vacuum and asymptotic properties of the kernel K, in the IR this integral is equivalent to 2d Euclidean YM theory with 2d coupling $g_{2D}^2 \equiv m g_{YM}^2$

- This means, in particular, that large spatial Wilson loops obey area law with string tension

$$\sqrt{\sigma} \;\simeq\; \sqrt{\frac{\pi}{2}}\, m$$

- Also, elementary $\langle B^a(x)\, B^b(y) \rangle$ correlator is

$$\langle B^a(x)\, B^b(y) \rangle \;\sim\; \delta^{ab} K^{-1}(|x-y|)$$

Inverse Kernel

- Using the standard Bessel function identities we may expand

$$\frac{J_1(u)}{J_2(u)} = \frac{4}{u} + 2u\sum_{n=1}^{\infty}\frac{1}{u^2-\gamma_{2,n}^2}$$

  where the $\gamma_{2,n}$ are the ordered zeros of $J_2(u)$.

- Inverse kernel is thus ($L \cong p^2/4m^2$)

$$K^{-1}(p) = 1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\vec{p}^2}{\vec{p}^2+M_n^2} \qquad M_n = \frac{\gamma_{2,n}m}{2}$$

- M_n can be interpreted as constituents out of which glueball masses are constructed

$$M_1 = 2.568m \qquad M_2 = 4.209m \qquad M_3 = 5.810m$$

Inverse Kernel (cont'd.)

- At asymptotically large spatial separations $|x-y| \to \infty$ inverse kernel takes the form

$$K^{-1}(|x-y|) \approx -\frac{1}{4\sqrt{2\pi|x-y|}} \sum_{n=1}^{\infty} (M_n)^{\frac{3}{2}} e^{-M_n|x-y|}$$

Glueball masses

- To find glueball states of given space-time quantum numbers, we compute equal-time correlators of invariant probe operators with appropriate J^{PC}
- For example, for 0^{++} states we take $tr(B^2)$ as a probe operator and compute

$$\langle tr(B^2)_x \, tr(B^2)_y \rangle \sim K^{-2}(|x-y|)$$

- At large distance, we will find contributions of *single particle poles*

$$\langle tr(B^2)_x \, tr(B^2)_y \rangle \sim \frac{1}{|x-y|} \sum_{n,m=1}^{\infty} (M_n M_m)^{3/2} \, e^{-(M_n+M_m)|x-y|}$$

$$\begin{aligned} M_{0^{++}} &= M_1 + M_1 = 5.14m \\ M_{0^{++*}} &= M_1 + M_2 = 6.78m \\ M_{0^{++**}} &= M_1 + M_3 = 8.38m \\ M_{0^{++***}} &= M_1 + M_4 = 9.97m \end{aligned}$$

0^{++} Glueballs

- For 2+1 Yang-Mills, the "experimental data" consists of a number of lattice simulations, largely by M. Teper et al (hep-lat/9804008, hep-lat/0206027)

- The following table compares lattice results for 0^{++} glueball states with analytic predictions. All masses are in units of the square root of string tension

State	Lattice, $N \to \infty$	Sugra	Our prediction	Diff, %
0^{++}	4.065 ± 0.055	4.07(input)	4.098	0.8
0^{++*}	6.18 ± 0.13	7.02	5.407	12.5
0^{++**}	7.99 ± 0.22	9.92	6.716	16
0^{++***}	9.44 ± 0.38 [5]	12.80	7.994	15
0^{++****}	$--$	15.67	9.214	$--$

0^{++} Glueballs (cont'd.)

- There are no adjustable parameters in the theory; the ratios of $M_{0^{++}}$ to $\sqrt{\sigma}$ are *pure numbers*
- We are able to predict masses of 0^{++} resonances, as well as the mass of the lowest lying member
- Results for excited state masses differ at the 10-15% level from lattice simulations. A possible explanation of such discrepancy is that those states have not been correctly identified on the lattice.
- The table below gives an updated comparison with relabeled lattice data

State	Lattice, $N \to \infty$	Our prediction	Diff, %
0^{++}	4.065 ± 0.055	4.098	0.8
0^{++*}	6.18 ± 0.13	5.407	$--$
0^{++**}	6.18 ± 0.13	6.716	$--$
0^{++***}	7.99 ± 0.22	7.994	0.05
0^{++****}	9.44 ± 0.38	9.214	2.4

0^{--} Glueballs

- For 0^{--} glueballs we compute

$$\langle \mathrm{Tr}\,(\bar\partial J \bar\partial J \bar\partial J)_x \, \mathrm{Tr}\,(\bar\partial J \bar\partial J \bar\partial J)_y \rangle \sim \frac{1}{64(2\pi|x-y|)^{\frac{3}{2}}} \sum_{n,\,m,\,k=1}^{\infty} (M_n M_m M_k)^{3/2} e^{-(M_n+M_m+M_k)|x-y|}$$

- Masses of 0^{--} resonances are the sum of three constituents : $M_n+M_m+M_k$

- The following table compares analytic predictions with available lattice data. All masses are in units of the $\sqrt{\sigma}$

State	Lattice, $N \to \infty$	Sugra	Our prediction	Diff,%
0^{--}	5.91 ± 0.25	6.10	6.15	4
0^{--*}	7.63 ± 0.37	9.34	7.46	2.3
0^{--**}	8.96 ± 0.65	12.37	8.73	2.5

Spin–2 States

- Similarly, analytic predictions for $2^{\pm+}$ states are compared with existing lattice data in the table above
- By *parity doubling*, masses of J^{++} and J^{-+} resonances should be the same which is not the case with lattice values for 2^{++*} and 2^{-+*}. This indicates that apparent 7-14% discrepancy may be illusory.
- An updated comparison with relabeled lattice data is given in the table below

State	Lattice, $N \to \infty$	Our prediction	Difference, %
2^{++}	6.88 ± 0.16	6.72	2.4
2^{-+}	6.89 ± 0.21	6.72	2.5
2^{++*}	8.62 ± 0.38	7.99	7.6
2^{-+*}	9.22 ± 0.32	7.99	14
2^{++**}	10.6 ± 0.7 [6]	9.26	13
2^{++***}	$--$	10.52	$--$

State	Lattice, $N \to \infty$	Our prediction	Difference, %
2^{++}	6.88 ± 0.16	6.72	2.4
2^{++*}	8.62 ± 0.38	7.99	7.6
2^{++**}	9.22 ± 0.32	9.26	0.4
2^{++***}	10.6 ± 0.7	10.52	0.8

Spin–2 States (cont'd.)

- Finally, the table below summarizes available lattice data for $2^{\pm-}$ states and compares it to analytic predictions

State	Lattice, $N \to \infty$	Our prediction	Difference, %
2^{+-}	8.04 ± 0.50	8.76	8.6
2^{--}	7.89 ± 0.35	8.76	10.4
2^{+-*}	9.97 ± 0.91	10.04	0.7
2^{--*}	9.46 ± 0.66	10.04	5.6

Higher Spin States and Regge Trajectories

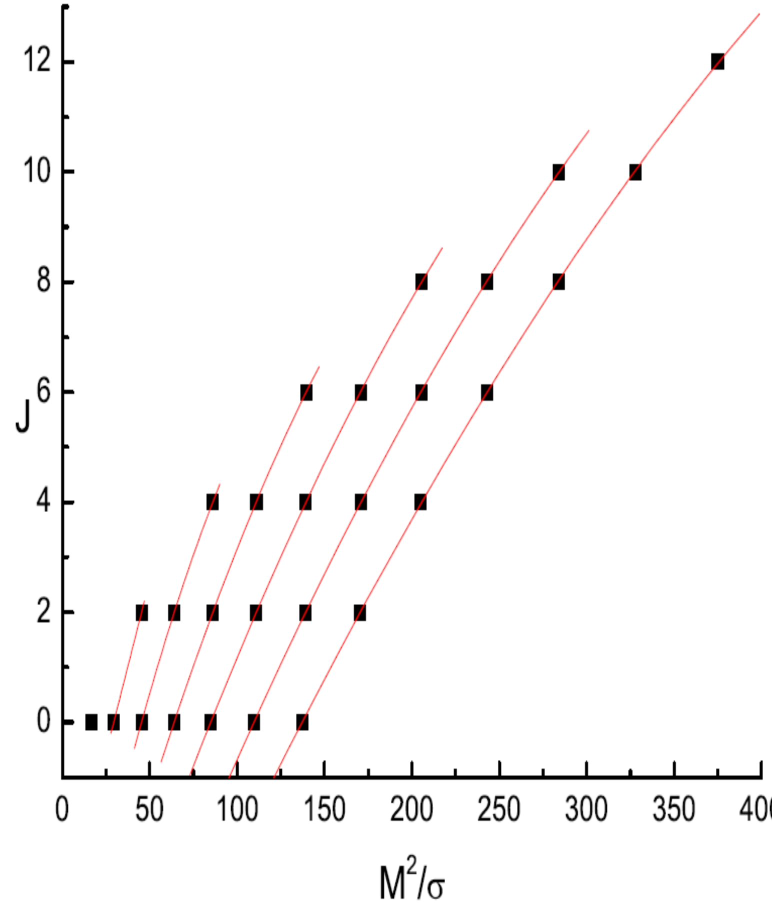
- It is possible to generalize our results for higher spin states
- For example, the masses of J^{++} resonances with even J are

$$M_{J^{++*n}} = M_{J/2+1} + M_{J/2+1+n}$$

- Similarly, the masses of J^{--} resonances with even J are

$$M_{J^{--*n}} = M_1 + M_{J/2+1} + M_{J/2+1+n}$$

- It is possible to draw nearly linear Regge trajectories.
  - Graph on the right represents a Chew-Frautschi plot of large N glueball spectrum. Black boxes correspond to J^{++} resonances with even spins up to J=12

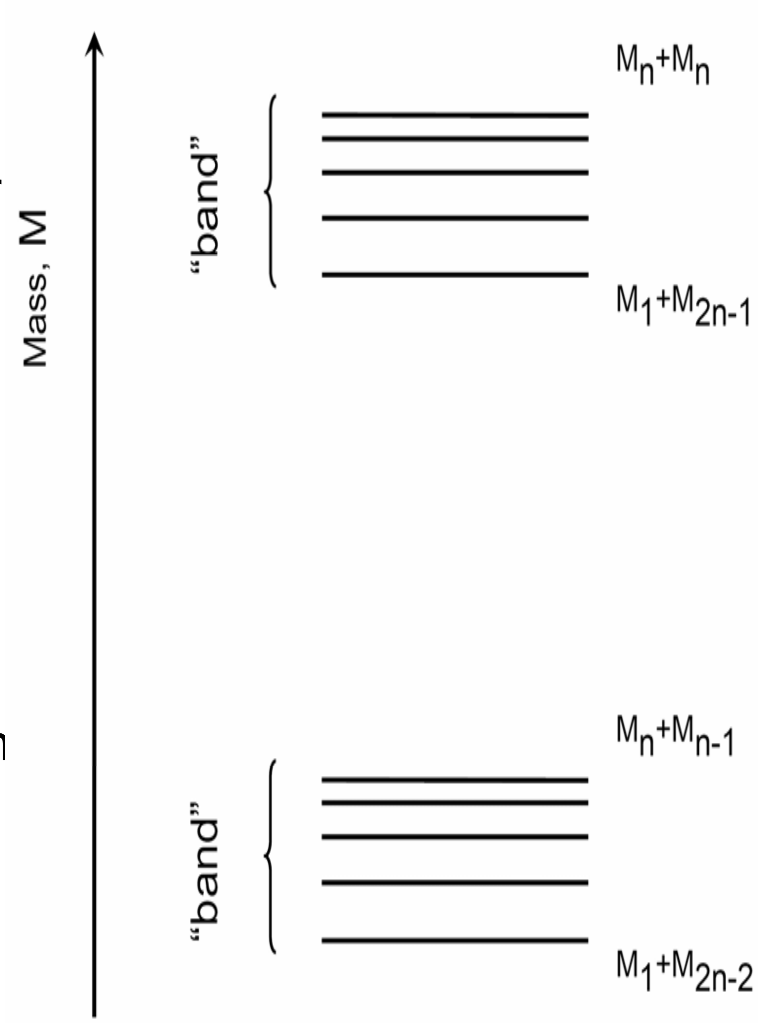


Approximate Degeneracy of Mass Spectrum

- The Bessel function is essentially sinusoidal and so its zeros are approximately evenly spaced (better for large n)
- Thus, the predicted spectrum has approximate degeneracies, e.g.

$$
\begin{aligned}
M_{0^{++**}} &= M_1 + M_3 = 8.38m \\
M_{2^{++}} &= M_2 + M_2 = 8.42m
\end{aligned}
$$

- The spectrum is organized into "bands" concentrated around a given level (which are well separated)



Outlook

- Results are very encouraging but many open questions remain
- Extensions in (2+1)d:
  - Add matter – meson spectrum
  - $1/N_c$ corrections
- Extension to (3+1)-dimensional YM
  - It is possible to generalize KKN (I. Bars) formalism to 3+1 dimensions: L. Freidel, hep-th/0604185.