Reconstucting Mass and Spin at Colliders

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Pheno 2006, Madison, May 15, 2006

How do we measure masses and spin?

Given a set of events $E(p_i^\mu)$,

- Take p_i^μ and directly construct (histogram) the observable for each event.

- If the observable requires some missing p_m^μ, construct a different observable $y(p_l^\mu)$ that requires only the measured p_l^μ.

- Hope (or discover by Monte Carlo) that $y(p_l^\mu)$ is correlated with "interesting" observable.

- Hope (or discover by Monte Carlo) that missing E_0, p_{z0} don't smear out signal (at hadron colliders).

Drawbacks to "classical" techniques

When missing energy is present or initial E_0 and p_{0z} are not known,

- Many observables to choose from. It's not obvious which is best.

- It's not obvious that any given observable is "optimal". One generally does not use all of the observed p_l^μ. There may be more information contained in the momenta that aren't used to construct observable $y(p_l^\mu)$.

- Many observables are constructed by finding kinematic edges, or "shapes". It's not obvious which shape to fit, whether background and combanitorics will smear out the shape.

- What will be the rate of "false positive" shape fits?

Classical Attempts

Many variables, many shapes, many edges, how does the experimentalist chose?

From Meade, Reece, **hep-ph/0601124**

$\langle H_t \rangle$: red

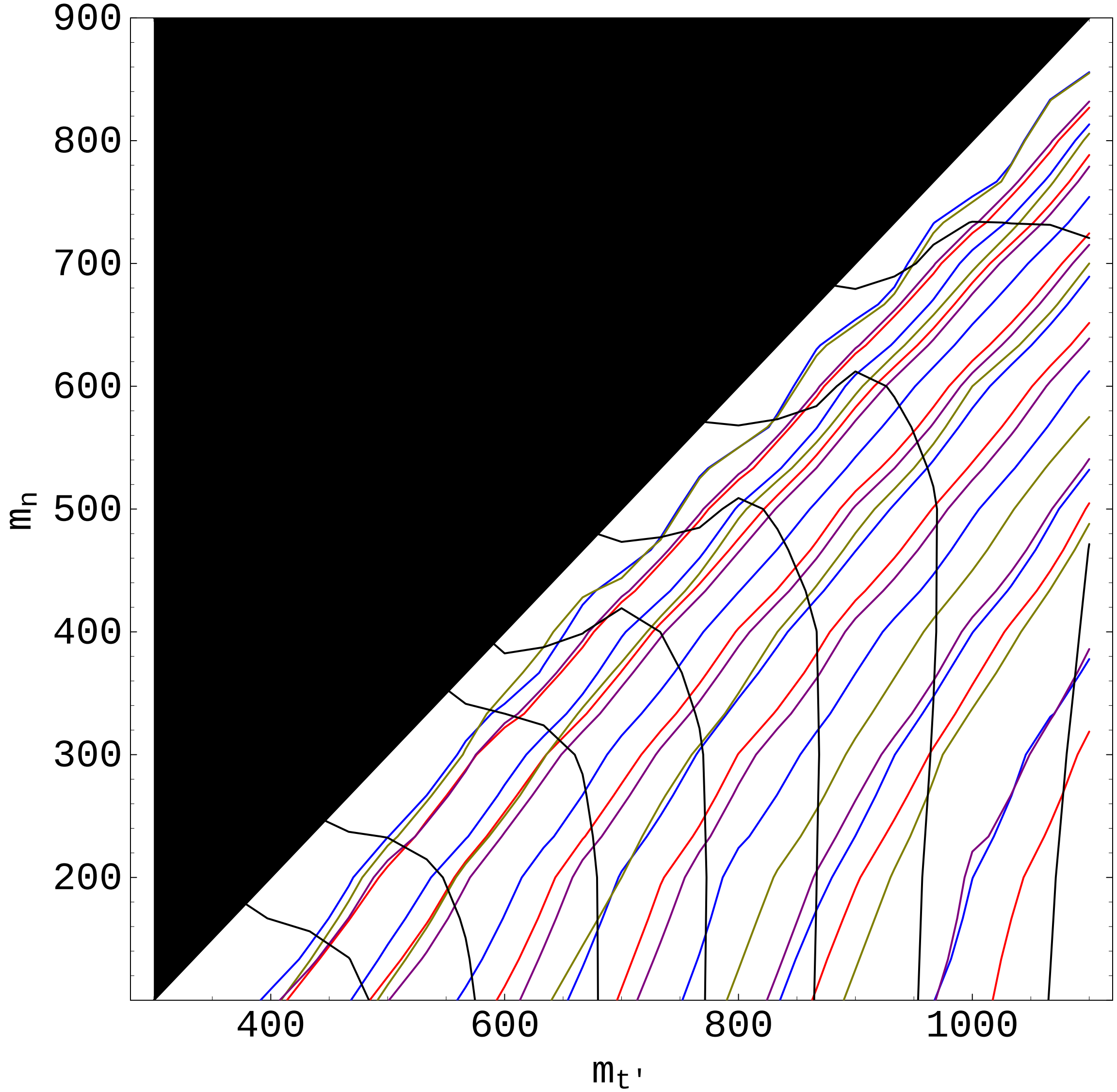
$\langle |\not{E}_T| \rangle$: blue

$\langle M_{\text{eff}} \rangle$: purple

M_{T2}^{max}: gold

σ: black

Mass Degeneracies are not fundamental (Alves, Éboli, Plehn, **hep-ph/0605067**)



Cross Sections as Probability Densities

A cross section generally is given by

$$\sigma = \frac{1}{F}\int \left|\mathcal{M}(p_0^\mu, p_i^\mu)\right|^2 \left(\prod_i \frac{d^3\vec{p}_i}{(2\pi)^3 2E_i}\right)(2\pi)^4\delta^4(p_0^\mu - \textstyle\sum_i p_i^\mu) \qquad (1)$$

for some initial state momenta p_0^μ and final state momenta p_i^μ. This is a zero-dimensional projection of a high-dimensional phase space, and as such contains very little information! Buried in here somewhere is all the information that is to be had.

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for some initial state momenta p_0^μ and final state momenta p_i^μ. This is a zero-dimensional projection of a high-dimensional phase space, and as such contains very little information! Buried in here somewhere is all the information that is to be had.

Let us do a little rearrangement to retain all information in the high-dimensional space.

$$P(p_i^\mu) = \frac{1}{\sigma} \frac{d\sigma}{\prod_i d^3\vec{p}} = \frac{(2\pi)^{4-3N}}{2^N F \sigma \prod_i E_i} \left|\mathcal{M}(p_0^\mu, p_i^\mu)\right|^2 \delta^4 \left(p_0^\mu - \sum_i p_i^\mu\right). \qquad (2)$$

this is a *probability density* expressing the probability of a particular configuration of momenta. For N external particles, it is a $3N - 4$ dimensional space.

Cross Sections as Probability Densities II

$$P(p_i^\mu) = \frac{1}{\sigma}\frac{d\sigma}{\prod_i d^3\vec{p}} = \frac{(2\pi)^{4-3N}}{2^N F\sigma \prod_i E_i}\left|\mathcal{M}(p_0^\mu, p_i^\mu)\right|^2 \delta^4\left(p_0^\mu - \sum_i p_i^\mu\right). \qquad (3)$$

In principle, one could directly compare this PDF (*Probability* Density Function) between simulated events and data. But, high-dimensional spaces require a lot of data to map out.

So, let us project this PDF onto a lower dimensional space.

$$P(E_1) = \int \frac{P((E_1; p_{1x}, p_{1y}, \sqrt{E_1^2 - m_1^2 - p_{1x}^2 - p_{1y}^2}), p_i^\mu)}{2(2\pi)^3 F\sigma\sqrt{E_1^2 - m_1^2 - p_{1x}^2 - p_{1y}^2}} \times dp_{1x}dp_{1y}\prod_{i\neq 1} d^3\vec{p}_i$$

where we have changed variables $p_{1z} = \sqrt{E_1^2 - m_1^2 - p_{1x}^2 - p_{2x}^2}$.

In this way we can obtain the *shape* of any distribution. All one-dimensional variables can be obtained in this manner, by performing an appropriate projection.

Overall normalization (here: σ) is always hard, but we don't need it!

Probability Densities for Hadron Colliders

The previous equations assumed all initial and final state momenta were known. e.g. a lepton collider. At hadron or photon colliders this is not the case. So we must integrate over the initial state as well.

$$P_{had}(p_i^\mu, x_1, x_2) = \frac{1}{\sigma}\frac{d\sigma}{dx_1 dx_1 \prod_i d^3\vec{p}}$$
$$= \frac{(2\pi)^{4-3N}}{2^N F\sigma \prod_i E_i} f_{i1}(x_1) f_{j2}(x_2) \left|\mathcal{M}_{ij}(p_0^\mu, p_i^\mu)\right|^2 \delta^4\left(p_0^\mu - \sum_i p_i^\mu\right).$$

for parton i and j having Parton Density Functions f_{i1} and f_{j2} respectively and $p_0^\mu = \sqrt{s}(x_1 + x_2; 0, 0, x_1 - x_2)$.

Probability Densiities with Missing Energy

If one expects new physics to explain the Dark Matter component of the universe, one generically expects a dark matter particle, with non-zero mass to escape the detector.

Therefore in events with missing particles, we must *project* the previous probability densities onto the space of measured particles

$$P_{\text{meas}}(p_l^\mu) \;=\; \frac{(2\pi)^{4-3N}}{2^N F\sigma \prod_l E_l} \int \left|\mathcal{M}_{ij}(p_0^\mu, p_l^\mu, p_m^\mu)\right|^2 \delta^4\left(\sum_i p_i^\mu\right) \prod_m \frac{d^3\vec{p}_m}{E_m}$$

for lepton colliders or

$$P_{\text{meas,had}}(p_l^\mu) = \frac{(2\pi)^{4-3N}}{2^N F\sigma \prod_l E_l} \times \int f_{1i}(x_1) f_{2j}(x_2) \left|\mathcal{M}_{ij}(p_0^\mu, p_l^\mu, p_m^\mu)\right|^2 \delta^4\left(\sum_i p_i^\mu\right) dx_1 dx_2 \prod_m \frac{d^3\vec{p}_m}{E_m}$$

for hadron or photon colliders.

Observables from PDF's with Missing Energy

Any observables y_k are a projection of $P_{\text{meas}}(p_i^\mu)$ onto a lower dimensional space of interest. One first changes variables from missing momentum components p_m^μ to the observables y_k, and integrates out the remaining momenta. With K observables and M missing momenta

$$P(y_k) = \frac{(2\pi)^{4-3N}}{2^N F\sigma} \int J(y_k(p_i^\mu)) f_{1i}(x_1) f_{2j}(x_2) \left|\mathcal{M}_{ij}(p_i^\mu)\right|^2 \delta^4\left(\textstyle\sum_i p_i^\mu\right) dx_1 dx_2 \prod_i \frac{1}{E_i} \prod_{n=1}^{3M-K} dp_n \prod_l d^3\vec{p}_l$$

for visible particles l and invisible particles m, where $P(p_i^\mu)$ is the full probability density (missing stuff too).

However nature samples this distribution for us, discretely. Therefore

$$P(y_k) = \frac{1}{\#\text{events}} \sum_{events} \frac{(2\pi)^{4-3N}}{2^N F\sigma \prod_l E_l}$$
$$\times \int J(y_k(p_i^\mu)) f_{1i}(x_1) f_{2j}(x_2) \left|\mathcal{M}_{ij}(p_i^\mu)\right|^2 \delta^4\left(\textstyle\sum_i p_i^\mu\right) dx_1 dx_2 \prod_m \frac{1}{E_m} \prod_{n=1}^{3M-K} dp_n$$

To Jacobian or not to Jacobian

In general this procedure involves constructing the (possibly very complicated) Jacobian of the variable transformation.

Jacobians, in general, may be complicated, non-analytic, multiple valued, and involve combanitorics (and therefore be a sum of several Jacobians).

So instead consider

$$P(y_k) = \frac{1}{\#\text{events}} \sum_{events} \int \left|\mathcal{M}_{ij}(p_i^\mu)\right|^2 Q(p_i^\mu) \prod_{k=1}^{3M-K} dp_k$$

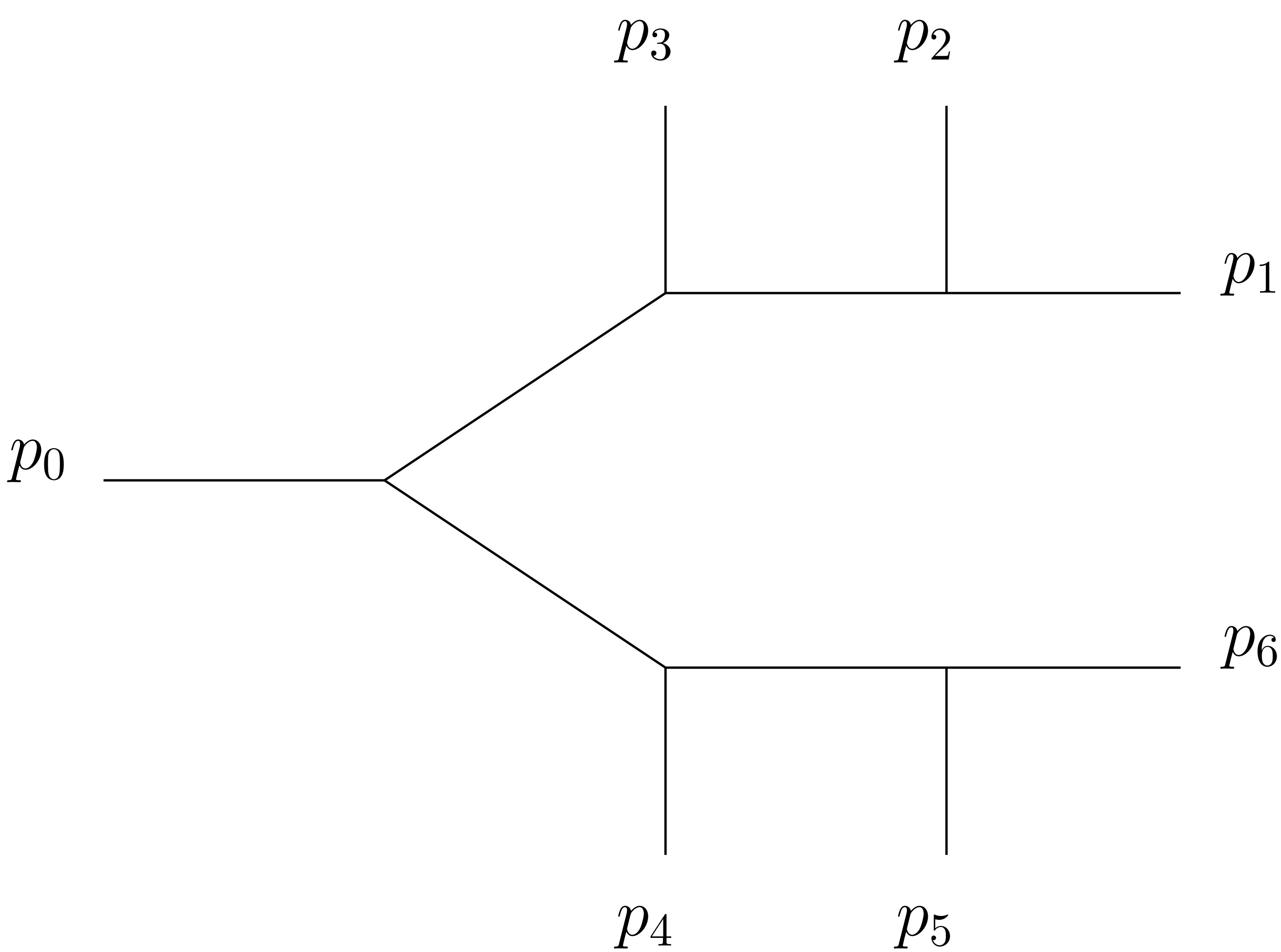
where $Q(p_i^\mu)$ is some arbitrary function. I've absorbed the parton density functions, all constants, and energies into $Q(p_i^\mu)$.

$Q(p_i^\mu)$ destroys your ability to interpret $P(y_k)$ as a physical probability distribution.

But! One can still compare $P(y_k)$ between theory and experiment.

$t\bar{t}$ Di-lepton Topology

We choose the $t\bar{t}$ di-lepton topology to study. *Many* SUSY process fit into this topology



Constraints

In order to use this technique, one must specify a hypothesis $\mathcal{M}(p_i^\mu)$. We take this to be

$$\begin{aligned}\mathcal{M}(p_i^\mu) &= \delta(p_1^2 - p_6^2) \\ &\times\delta((p_1+p_2)^2 - (p_5+p_6)^2) \\ &\times\delta((p_1+p_2+p_3)^2 - (p_4+p_5+p_6)^2)\end{aligned}$$

i.e. Narrow Width Approximation.

If we take E_1 and E_6 to be free (i.e. we don't know the missing particle's mass), there are 3 missing momenta to integrate. Choose E_1, E_6, p_{6z}.

E_0, p_{0z}	2
p_1^μ, p_6^μ	8
$\sum_i p_i^\mu = 0$	-4
$\mathcal{M}(p_i^\mu)$ constraints	-3
variables to integrate	3

These constraints remove a significant amount of background and combanitorics!

Project into Mass space

Now, project into Mass space and construct $P(M_1, M_2, M_3)$. We can project onto at most 3 dimensions with this choice of $\mathcal{M}(p_i^\mu)$.

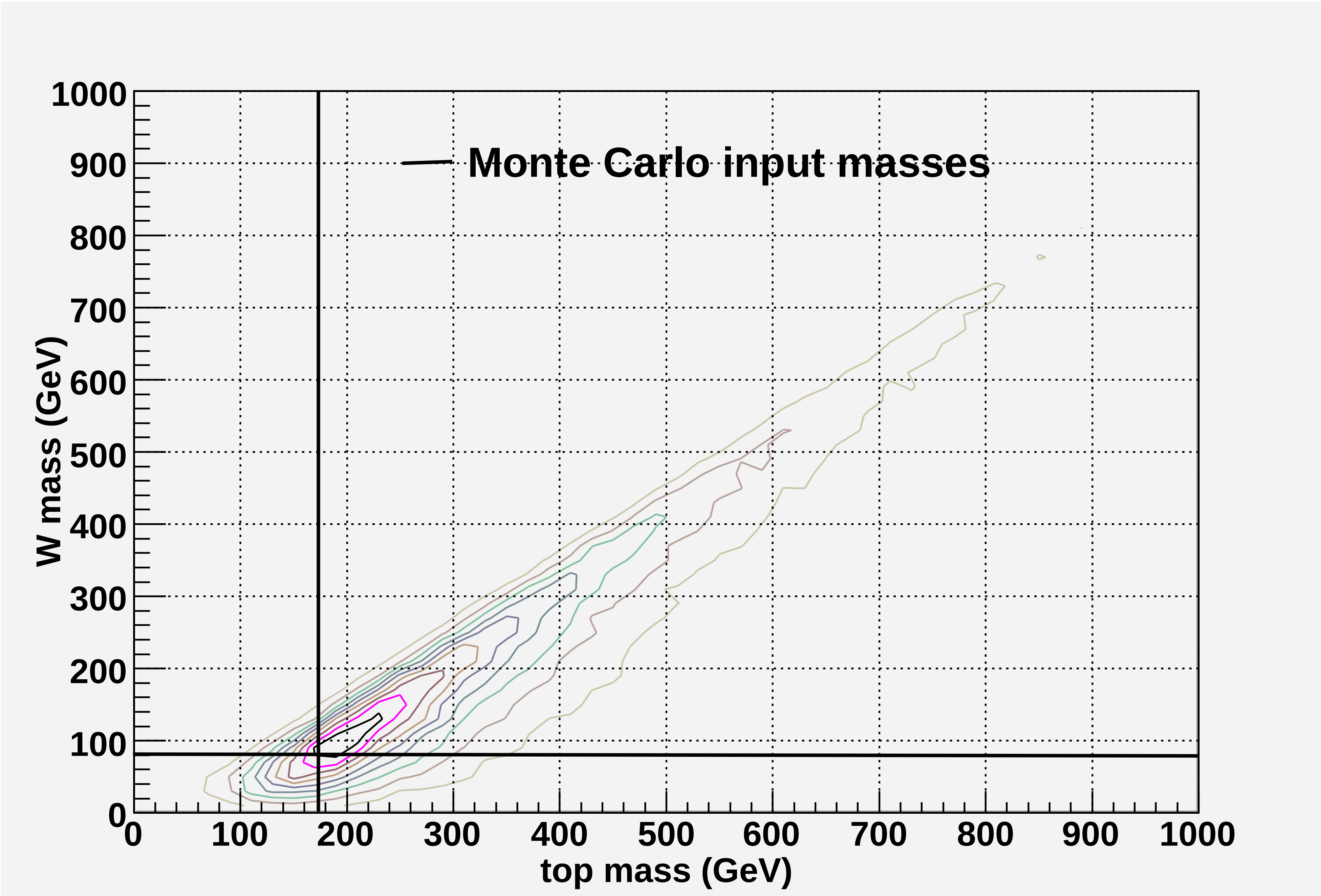
The constraints give complicated, multiple valued solutions. Therefore we will not attempt to construct a Jacobian.

Since the integral must be done using monte carlo techniques, we must specify a PDF from which to sample the quantities being integrated. But that PDF is a function of the masses, and we don't know the masses yet!

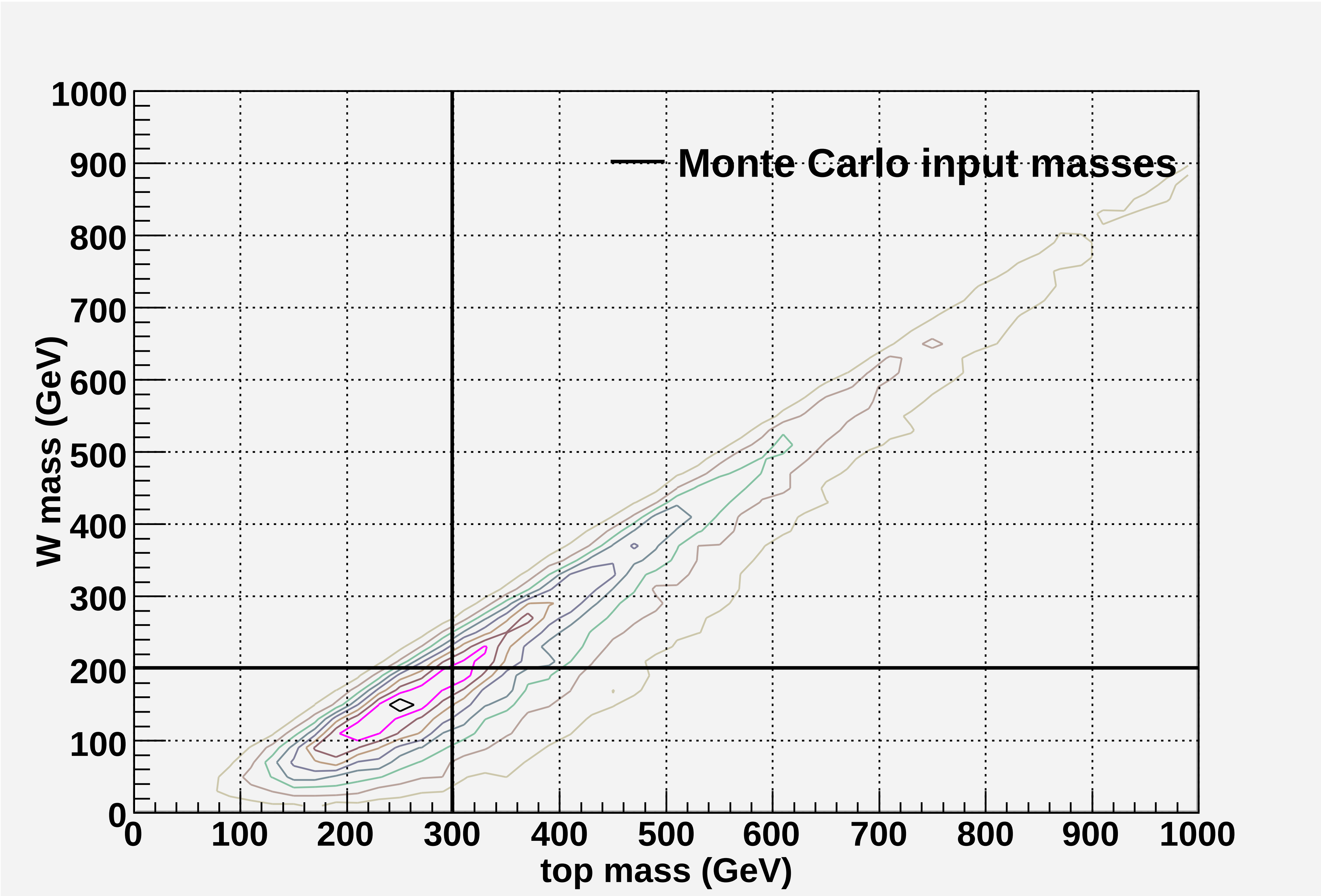
Therefore we will choose something simple (e.g. uniform on $[0, 14\text{TeV}]$)

$Q(p_i^\mu)$ is a computer subroutine that fills a histogram of the masses.

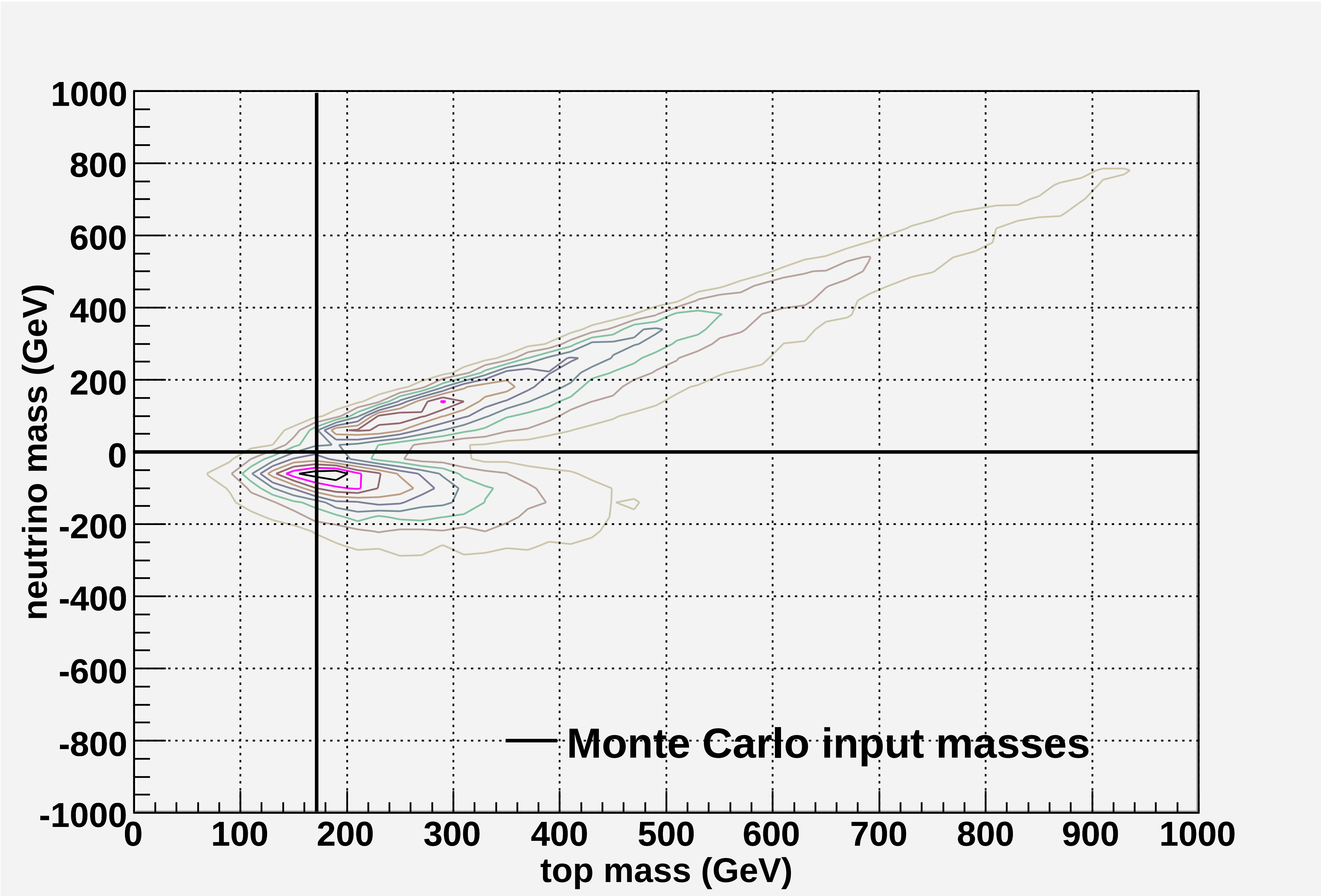
This means $P(M_1, M_2, M_3)$ is *not* a probability density, but just some transformation on the kinematics of the event. (But we know it has something to do with mass!)

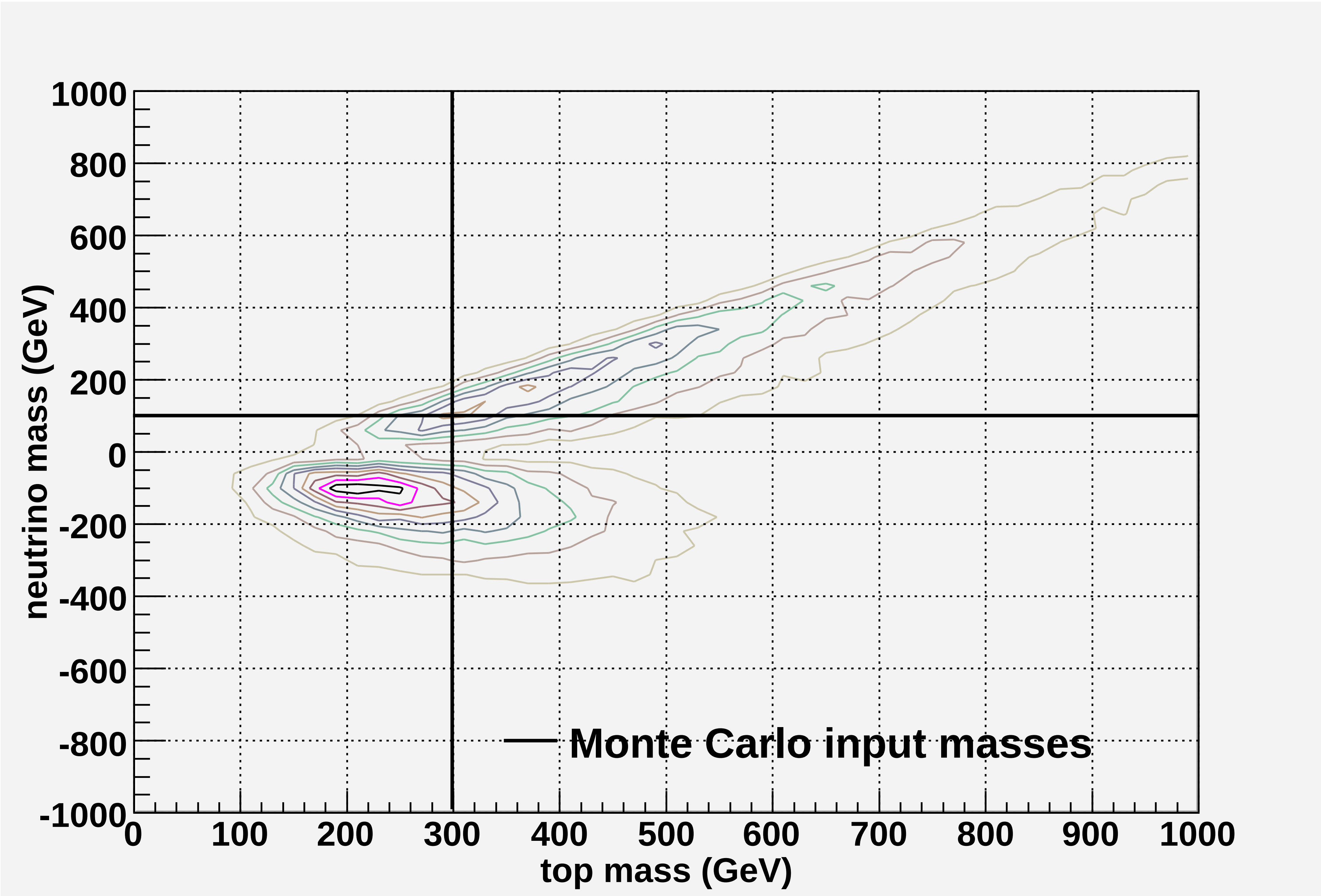


Monte Carlo input masses
W mass (GeV)
top mass (GeV)
0
100
200
300
400
500
600
700
800
900
1000



Monte Carlo input masses
W mass (GeV)
top mass (GeV)
0
100
200
300
400
500
600
700
800
900
1000


1000
800
600
400
200
0
-200
-400
-600
-800
-1000
0
100
200
300
400
500
600
700
800
900
1000
neutrino mass (GeV)
top mass (GeV)
Monte Carlo input masses



1000
800
600
400
200
0
-200
-400
-600
-800
-1000
neutrino mass (GeV)
0
100
200
300
400
500
600
700
800
900
1000
top mass (GeV)
Monte Carlo input masses

Why is there no mass scale degeneracy

The first kinematic quantity I solve for is p_{1z}, which is a quadratic

$$p_{1z}^2 + Ap_{1z} + (E_6^2 - p_{6z}^2 - E_1^2 + BE_6 + Cp_{6z} + DE_1 + F) = 0. \quad (4)$$

This has a discriminant

$$D_{p_{1z}} = A^2 - 4(E_6^2 - p_{6z}^2 - E_1^2 + BE_6 + Cp_{6z} + DE_1 + F) \geq 0. \quad (5)$$

Therefore one can see a degeneracy for $E_1 \simeq E_6 \to \infty; p_{6z} \simeq 0$

The ability to get a positive discriminant provides significant background and combanitoric rejection!

Even with a positive $D_{p_{1z}}$, most solutions have *negative* M^2.

One can derive a condition that $M_W^2 = 0$. This condition is a quartic and has a mass degeneracy for

$$-(Q_1^2 - I_1^4)E_6^4 + (P_1^2 - I_1^2)E_1^4 + (R_1^2 - G_1^4)p_{6z}^4 \simeq 0. \quad (6)$$

Why is there no mass scale degeneracy, ctd.

Therefore, the mass scale degeneracy represents a *lower dimensional* subspace of the Probability Density Function. Therefore it has no *volume* in the higher dimensional space, and has no probability!

The intersection of the $M^2 > 0$ and $D_{p_{1z}} \geq 0$ conditions gives an *even lower* dimensional space.

Therefore, it is possible to make 1-dimensional projections that will show mass scale degeneracies, since for a given event, there generally *is* a 1-dimensional degeneracy. But this is an un-clever choice of variables.

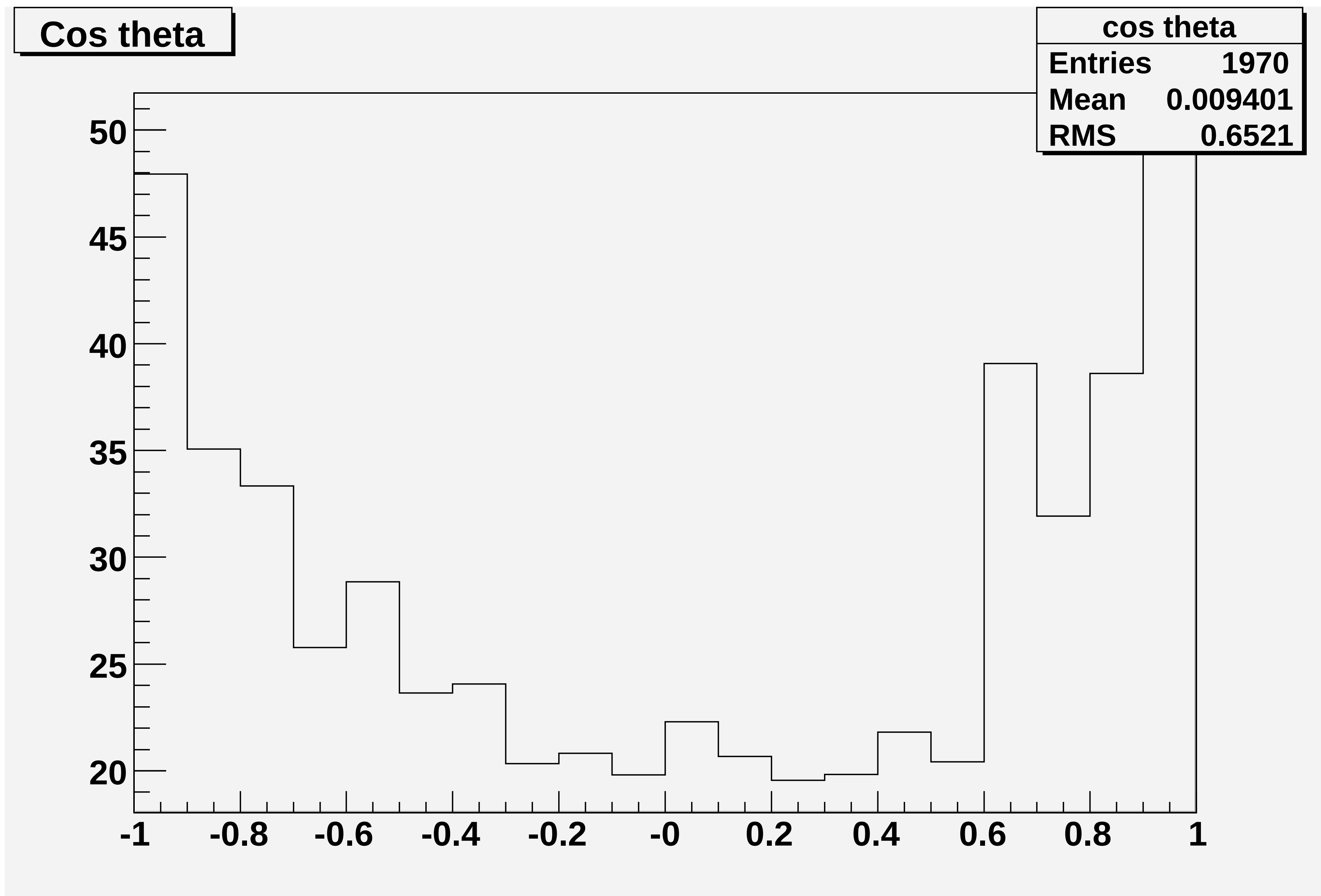
Spin

Projecting onto spin is much easier.

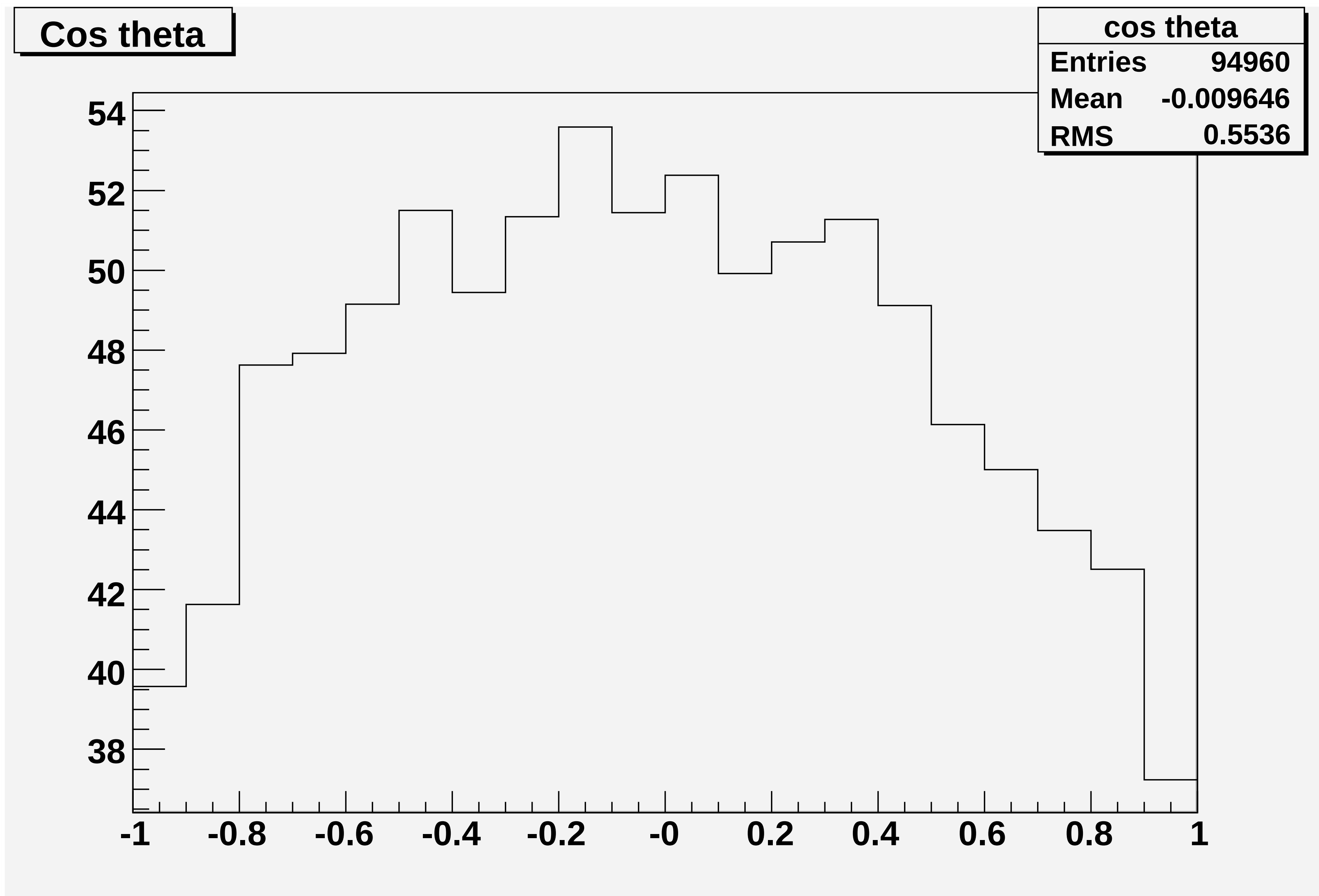
Using the same $\mathcal{M}(p_i^\mu)$, one can simply histogram any angle one desires. Since $\mathcal{M}(p_i^\mu)$ is not a function of any angles, angles are not affected by it. (Unlike masses)

One can histogram $\cos\theta$ for any subdiagram of the event, simultaneously fitting the spin of several particles.

It is necessary to have the correct masses!



Cos theta
cos theta
Entries 1970
Mean 0.009401
RMS 0.6521
50
45
40
35
30
25
20
-1
-0.8
-0.6
-0.4
-0.2
-0
0.2
0.4
0.6
0.8
1



Cos theta
cos theta
Entries 94960
Mean -0.009646
RMS 0.5536
54
52
50
48
46
44
42
40
38
-1
-0.8
-0.6
-0.4
-0.2
-0
0.2
0.4
0.6
0.8
1

Conclusions

The correct way to (analytically) obtain physical observables is to take the fully differential PDF $1/\sigma d\sigma/d\prod_i \vec{p}_i$ and project it onto the observable you're interested in by changing one momentum component to that variable, and integrating over all other momenta.

Missing particles must be *projected out* from the full PDF to obtain $P_{\text{meas}}(p_l^\mu)$.

The missing particle projection allows one to consider their distributions to be continuous, performing the projection on an event-by-event basis.

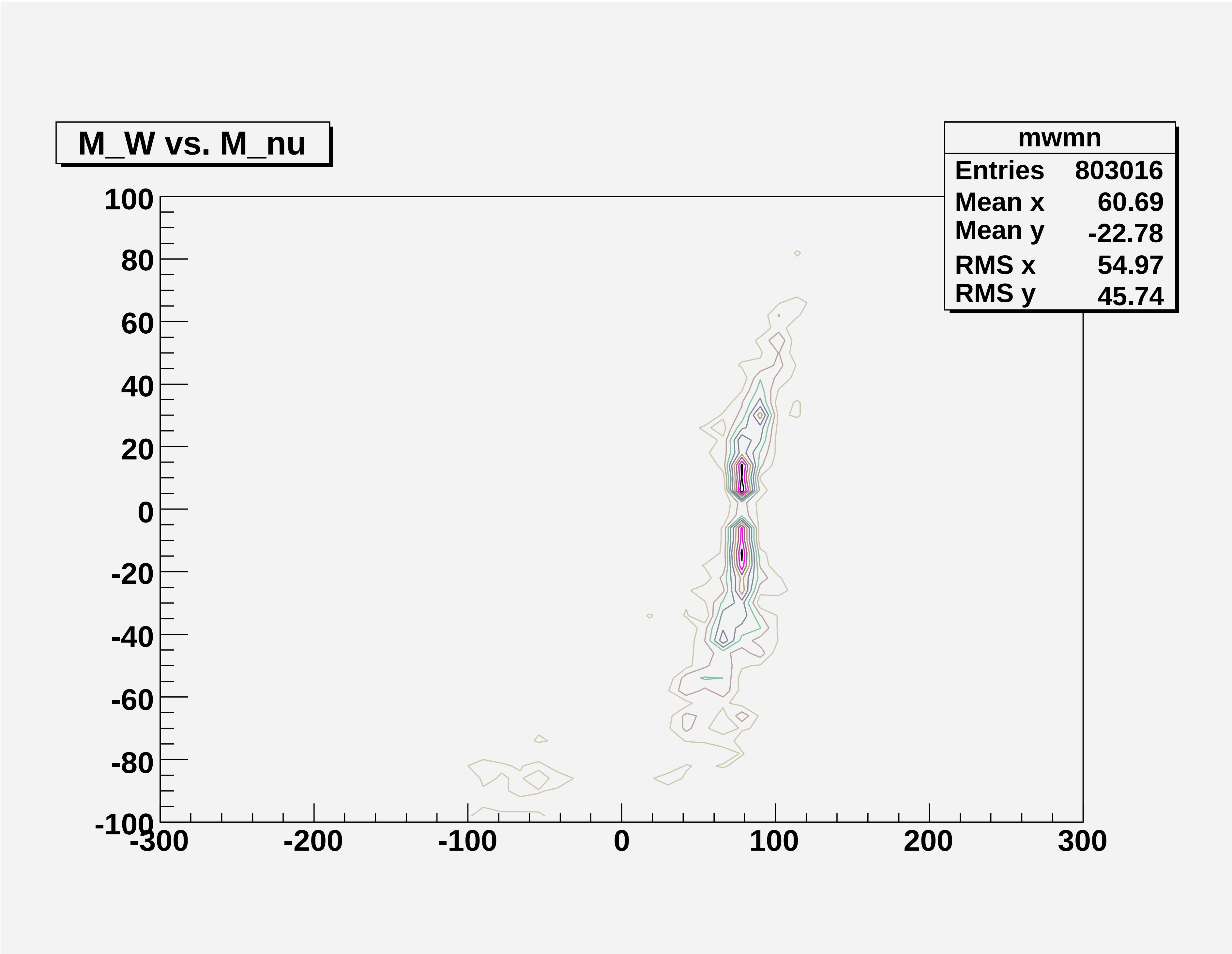
Events with > 2 missing particles become very hard.

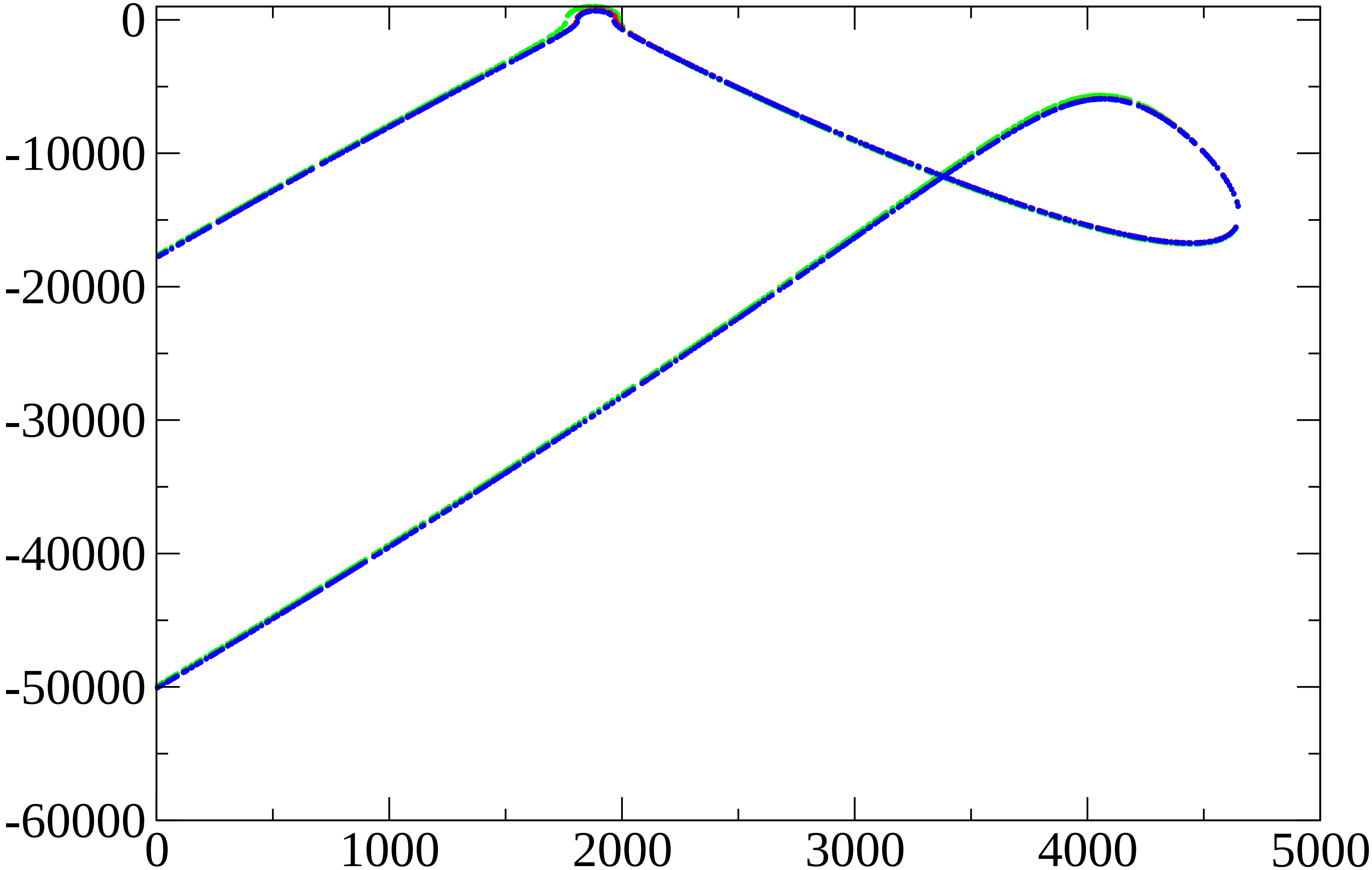
Events with more resonances (more constraints) can be totally solved using constraints. 2 missing particles with 4 intermediate masses at a hadron collider allows one to solve for *all* components of the neutrino missing momentum.

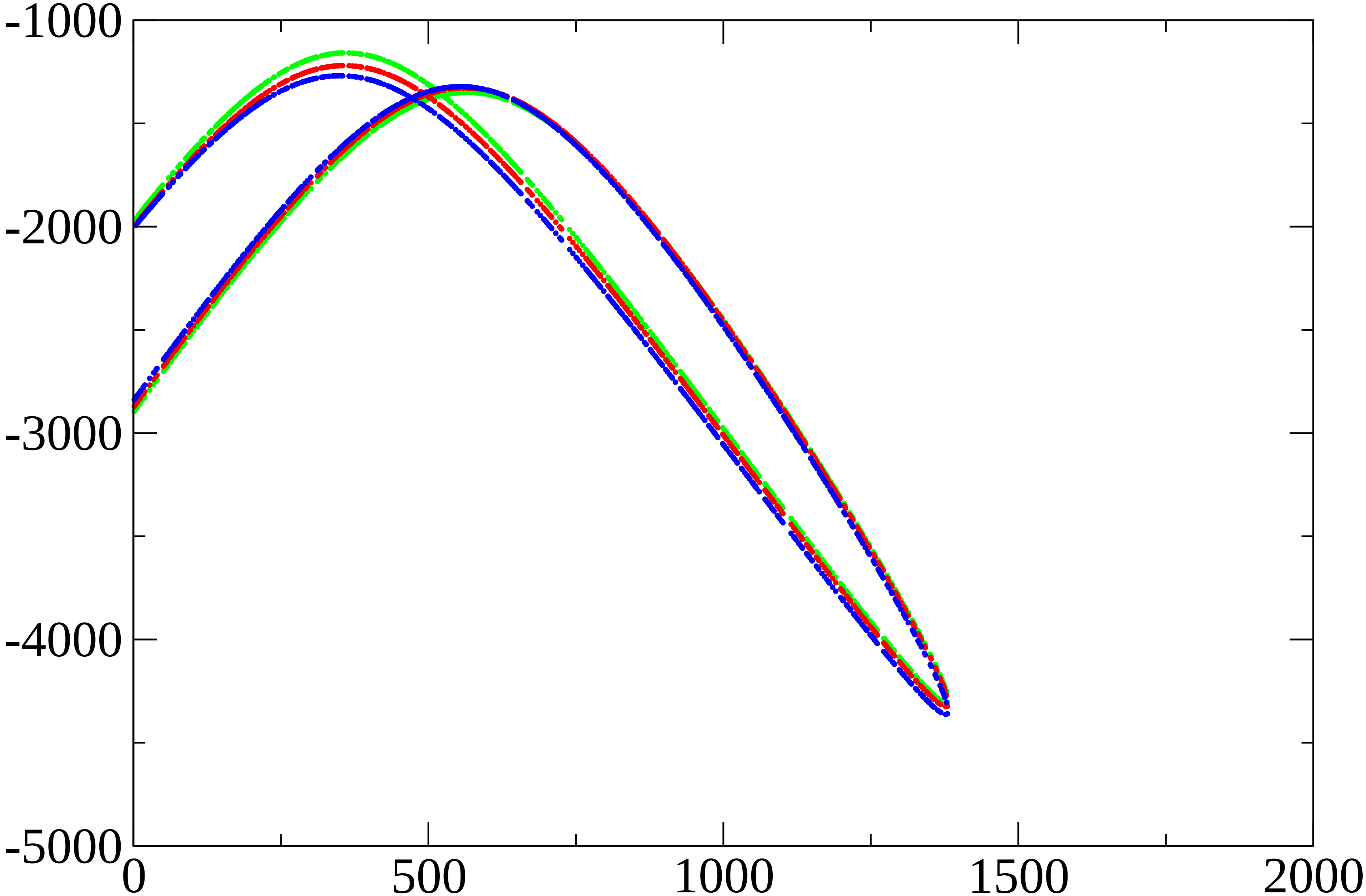
Conclusions

This eliminates the necessity to choose shapes and clever variables.

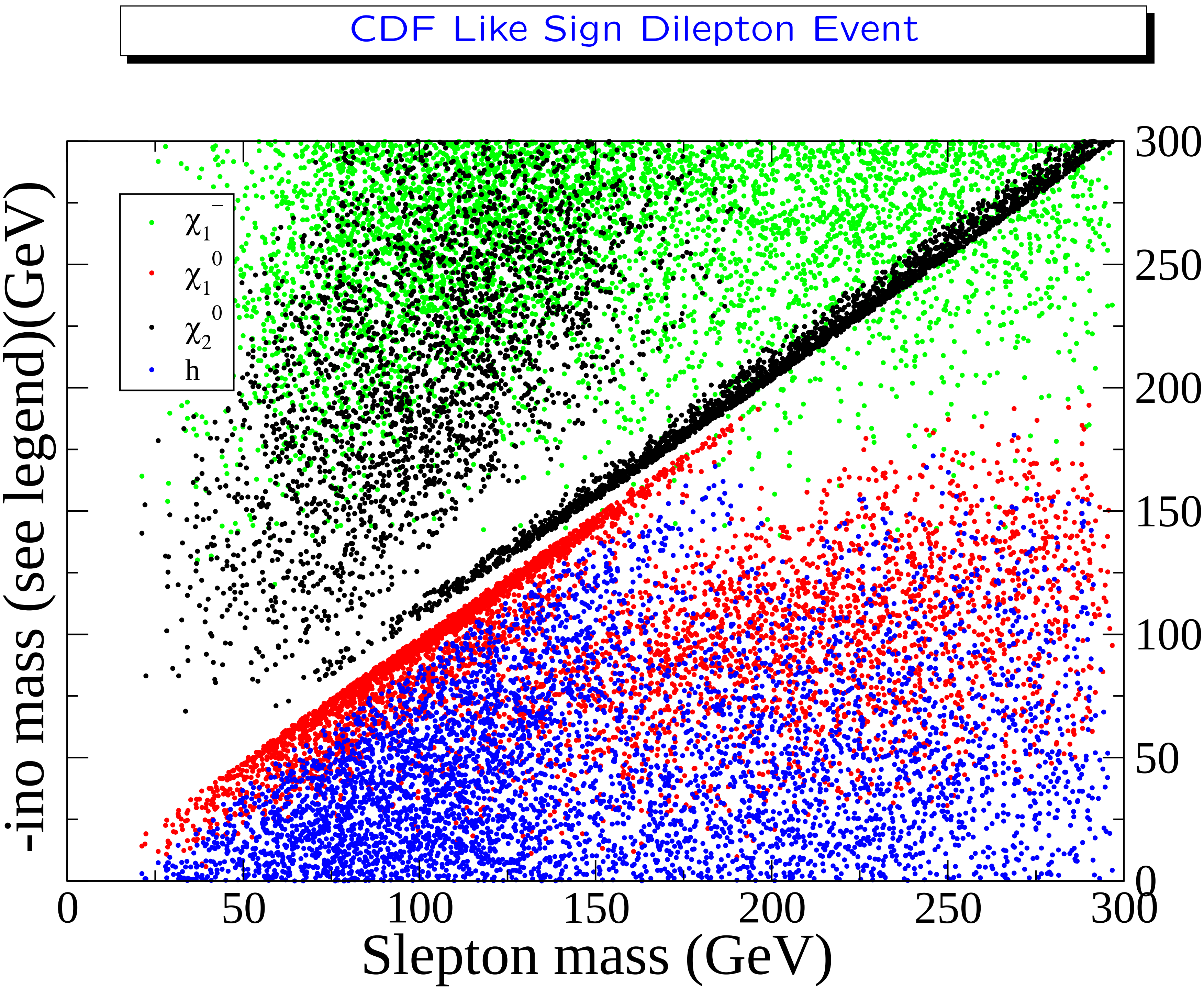
Simple Matrix element hypotheses can be tested in an automated way. Lagrangians can be built from the bottom up, one resonance at a time.


M_W vs. M_nu
mwmn
Entries 803016
Mean x 60.69
Mean y -22.78
RMS x 54.97
RMS y 45.74
100
80
60
40
20
0
-20
-40
-60
-80
-100
-300
-200
-100
0
100
200
300


0
-10000
-20000
-30000
-40000
-50000
-60000
0
1000
2000
3000
4000
5000



-1000
-2000
-3000
-4000
-5000
0
500
1000
1500
2000


CDF Like Sign Dilepton Event
χ_1^-
χ_1^0
χ_2^0
h
-ino mass (see legend)(GeV)
Slepton mass (GeV)
0
50
100
150
200
250
300