Five-loop QED contributions for the anomalous magnetic moment of the muon

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Introduction & Motivation

Generalities; Anomalous magnetic moment of the muon: a_μ

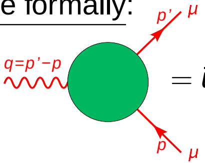
- Magnetic moment of any system:
  I.) Motion of el. charges   II.) Intrinsic mag. moments
- Dirac theory predicts for a lepton $\ell = e, \mu, \tau$:

$$\vec{\mu}_\ell = g_\ell \left(\frac{e}{2m_\ell} \right) \vec{S}, \quad g_\ell = 2$$

- Quantum fluctuations: $\rightsquigarrow$ deviation from $g_\ell = 2$:

$$g_\ell = 2(1 + a_\ell) \qquad \rightsquigarrow \text{precise test of QFT}$$

More formally:



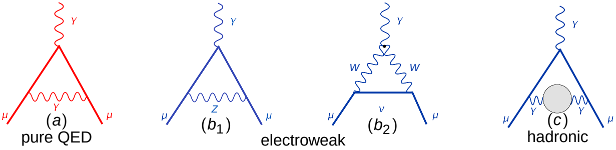
$$= \bar{u}(p') \left[\gamma^\mu F_E(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2m_\mu} F_M(q^2) \right] u(p)$$

- In the static limit$(q^2 \to 0)$: $F_E(0) = 1,$ $F_M(0) = a_\mu$

Introduction & Motivation

Theory: Higher order corrections

- Higher order corrections are classified into 3 classes:



- The QED part is known to 4-loops (and leading terms in 5 loops!)
  (next slide)
- The EW part is known to 2-loops
  R. Jackiw, S. Weinberg; G. Altarelli et al.; I. Bars, M. Yoshimura; A. Czarnecki et al.
- The hadronic part is known but with limited accuracy
  Bouchiat,et al.;M. Gourdin,et al.;Brodsky, de Rafael;Hagiwara et al.;Alemany et al.;Davier et al.;Passera et al.
  Dominant theoretical uncertainties to the muon anomaly
  Effects at present not from first principles

Introduction & Motivation

Theory: QED contributions to a_μ

$$a_\mu^{\text{QED}} = \left(\frac{\alpha}{\pi}\right) 0.5$$

Schwinger

$$+ \left(\frac{\alpha}{\pi}\right)^2 0.765857410(27)$$

Sommerfield; Petermann; Suura & Wichmann; Elend

$$+ \left(\frac{\alpha}{\pi}\right)^3 24.05050964(87)$$

Barbieri, Laporta, Remiddi,...; Kinoshita;
Czarnecki, Skrzypek; Friot, Greynat, de Rafael

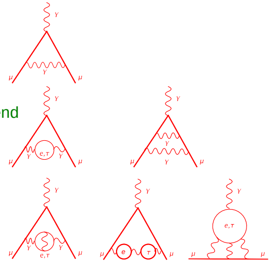
$$+ \left(\frac{\alpha}{\pi}\right)^4 130.8055(80)$$

Kinoshita, Lindquist; Kinoshita, Nio; Kinoshita, Nizic,
Okamoto; Aoyama, Hayakawa, Kinoshita, Nio; Lautrup, de Rafael

$$+ \left(\frac{\alpha}{\pi}\right)^5 663(20)$$

In progress

Kinoshita et al.; Kataev; Laporta; Baikov et al.



growing coefficients

Introduction & Motivation

The value

The present experimental value is terrifically accurate!:

$$a_\mu^{\mathrm{exp}} = 116592080(63) \cdot 10^{-11}$$

E821: Final Report: PRD73 (2006)

$\rightsquigarrow$ Improvement of a factor of 14 compared to the classic CERN experiment

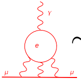
- QED four-loop contributes as much as $380.8 \cdot 10^{-11}$ (compared to the exp. uncertainty of $\sim 60 - 70 \cdot 10^{-11}$)
- [1] $$\Delta a_\mu = a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{theo.}} = 292(86) \cdot 10^{-11}$$ The current theory prediction shows an "interesting but not yet conclusive discrepancy" of $\sim 3\sigma$ (or $\sim 1\sigma$ if one uses τ-data to describe the hadronic effects)

[1] A. Höcker, W. Marciano, PDG

Calculation

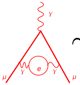
logarithmically enhanced contributions

There are 2 sources of numerically leading enhanced logs of large ratio $\frac{M_\mu}{m_e} = 206.7682838$ pure QED contributions:



$\sim \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{2\pi^2}{3} \ln\left(\frac{M_\mu}{m_e}\right) + \ldots\right),$

LBL: light by light scattering



$\sim \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{1}{3} \ln\left(\frac{M_\mu}{m_e}\right) + \ldots\right)$

VP: vacuum polarization

$\rightarrow$ We will consider mixed VP contributions: photon propagator composed from electron loops and photon exchanges only!

$$a_\mu = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \left[d_R \left(\frac{-x^2}{1-x} \frac{M_\mu}{m_e}, \alpha \right) - 1 \right]$$

B. Lautrup, de Rafael ’74

with $d_R(q^2/m^2, \alpha) = 1/(1 + \alpha \Pi^{\mathrm{OS}}(q^2/m_e^2, \alpha^{\mathrm{OS}}))$

Π^{OS} proper photon VP in OS-scheme

Calculation

Vacuum polarization function

Required ingredients for the calculation:

- Vacuum polarization function
  - High-energy limit $\to$ massless propagators
    The value of $\overline{\Pi}(Q^2, m = 0, \overline{\alpha})$ is known at 4-loops
    with Baicer (P. Baikov, 2000-2008) in $\overline{\text{MS}}$-scheme

    P.A. Baikov, K.G. Chetyrkin, J.H. Kühn

Traditionally in calculations of a_ℓ everybody uses the classical OS-scheme: α and all lepton masses are on-shell and:

$$\Pi^{\text{OS}}(Q = 0, m, \alpha^{\text{OS}}) = 0$$

$\rightsquigarrow$ Transform massless propagator from $\overline{\text{MS}} \to$ on-shell scheme

- Need: $\overline{\text{MS}} \leftrightarrow$ On-shell relation at 4-loop
  for fine structure constant conversion: $\overline{\alpha} \to \alpha^{\text{OS}}$

Calculation

On-shell $\leftrightarrow$ $\overline{\text{MS}}$-scheme for α

In QED MS in OS-schemes are related through:

$$\frac{\alpha^{\text{OS}}}{1+\Pi^{\text{OS}}(Q,m,\alpha^{\text{OS}})} = \frac{\overline{\alpha}}{1+\overline{\Pi}(Q,\overline{m},\overline{\alpha})}$$

scheme invariant concept of the invariant charge

Required: $\overline{\Pi}(Q=0,\overline{m},\overline{\alpha})$

$$\rightsquigarrow \overline{\alpha} = \alpha^{\text{OS}}\left(1+\sum_{i\geq 1} C^{(i)}_{\overline{\alpha}\alpha}\left(\frac{\alpha^{\text{OS}}}{\pi}\right)^i\right)$$

- IBP, 11 master integrals
- $\overline{\text{MS}}$ $\leftrightarrow$ On-shell relation at 3-loop for mass conversion
  Steinhauser, Chetyrkin; Melnikov, van Ritbergen

Results & Discussion

$\overline{\text{MS}}$-OS-relation for conversion of fine structure constant

$$\overline{\alpha} = \alpha^{\text{OS}} \left(1 + \sum_{i\geq 1} C^{(i)}_{\overline{\alpha}\alpha} \left(\frac{\alpha^{\text{OS}}}{\pi}\right)^i\right)$$

$$C^{(4)}_{\overline{\alpha}\alpha} = \frac{14327767}{9331200} + \frac{8791}{3240}\pi^2 + \frac{204631}{259200}\pi^4 - \frac{175949}{4800}\zeta_3 + \frac{1}{24}\pi^2\zeta_3 + \frac{9887}{480}\zeta_5 - \frac{595}{108}\pi^2\ln 2$$
$$- \frac{106}{675}\pi^4\ln 2 + \frac{6121}{2160}\pi^2\ln^2 2 - \frac{32}{135}\pi^2\ln^3 2 - \frac{6121}{2160}\ln^4 2 + \frac{32}{225}\ln^5 2 - \frac{6121}{90}a_4 - \frac{256}{15}a_5$$
$$+ \ell_{\mu m}\left[-\frac{383}{31104} + \frac{23}{108}\pi^2 - \frac{41}{144}\zeta_3 - \frac{2}{9}\pi^2\ln 2\right] + \frac{43}{144}\ell^2_{\mu m} + \frac{13}{108}\ell^3_{\mu m} + \frac{1}{81}\ell^4_{\mu m},$$

$$\ell_{\mu m} = \ln\frac{\mu}{m}, \quad a_n = \text{Li}_n\left(\frac{1}{2}\right)$$

Important: $\Pi(q^2/m^2, \alpha) = \Pi^\infty(q^2/m^2, \alpha) + \mathcal{O}(m^2/q^2)$
then the resulting error in a_μ will be:

$$a_\mu = \frac{\alpha}{\pi}\int_0^1 dx(1-x)\left[d_R^\infty\left(\frac{-x^2}{1-x}\frac{M_\mu}{m_e}, \alpha\right) - 1\right] + \mathcal{O}\left(\frac{m_e}{M_\mu}\right)$$

of order m_e/M_μ with $d_R^\infty = 1/(1+\alpha\Pi^\infty)$

Results & Discussion

Analytical 5-loop contributions to a_μ

The resulting contributions to a_μ coming from 4-loop terms in the photon propagator read :

$$a_\mu^{\text{asymp.}} = \sum_{i\geq 2} a_\mu^{\text{asymp.},(i)} \left(\frac{\alpha}{\pi}\right)^i$$

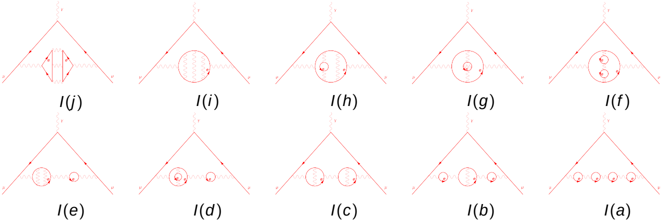
$$\begin{aligned} a_\mu^{\text{asymp.},(5)} = & -\frac{296496193}{41990400} + \frac{45709}{58320}\pi^2 + \frac{212701}{518400}\pi^4 - \frac{4488523}{259200}\zeta_3 + \frac{35}{144}\pi^2\zeta_3 + \frac{4}{3}\zeta_3^2 + \frac{10909}{720}\zeta_5 \\ & + \frac{35}{8}\zeta_7 - \frac{55}{24}\pi^2\ln 2 - \frac{53}{675}\pi^4\ln 2 + \frac{6121}{4320}\pi^2\ln^2 2 - \frac{16}{135}\pi^2\ln^3 2 - \frac{6121}{4320}\ln^4 2 \\ & + \frac{16}{225}\ln^5 2 - \frac{6121}{180}a_4 - \frac{128}{15}a_5 + \ell_{\mu e}\left[\frac{1416095}{279936} + \frac{41}{972}\pi^2 - \frac{1855}{432}\zeta_3 - \frac{10}{3}\zeta_5 - \frac{2}{9}\pi^2\ln 2\right] \\ & + \ell_{\mu e}^2\left[-\frac{1507}{1944} + \frac{8}{81}\pi^2 + \frac{4}{3}\zeta_3\right] - \frac{83}{243}\ell_{\mu e}^3 + \frac{8}{81}\ell_{\mu e}^4 + \mathcal{O}\left(\frac{m_e}{M_\mu}\right), \qquad \ell_{\mu e} = \ln\frac{M_\mu}{m_e}, \quad a_n = \text{Li}_n\left(\frac{1}{2}\right) \end{aligned}$$

Numerically: $\left(\frac{\alpha}{\pi}\right)^5 a_\mu^{asymp.,(5)} = \left(\frac{\alpha}{\pi}\right)^5 62.2667 = 0.42105 \cdot 10^{-11}$

(compared to $\left(\frac{\alpha}{\pi}\right)^5 663(20)$)

Comparison

Analytical ⟷ Numerical results



Subset	analytical	numerical	
$I(j)$	-1.21429	-1.24726(12)	✔
$I(i)$	+0.25237	–	
$I(g)+I(h)$	+1.50112	+1.56070(64)	✔
$I(f)$	+2.89019	+2.88598(9)	✔
$I(c)$	+4.81759	+4.74212(14)	✔
$I(d)$	+7.44918	+7.45270(88)	✔
$I(e)$	-1.33141	-1.20841(70)	✔
$I(b)$	+27.7188	+27.69038(30)	✔
$I(a)$	+20.1832	+20.14293(23)	✔

-Numerics from: T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, N. Watanabe (2008) M. Nio, T. Aoyama, M. Hayakawa, T. Kinoshita (2007,2008) and T. Kinoshita, M. Nio (2006)

-Remaining differences should come from power suppressed corrections to the asymptotic result of $\mathcal{O}(m_e/M_\mu)$

-Agreement with Kataev, where available

Summary & Conclusion

- Analytical and numerical methods help in computing the QED contribution to $(g-2)_\mu$
- The conversions formula for $\alpha^{OS}/\alpha^{\overline{MS}}$ is evaluated to four-loops $\Rightarrow$ one could reexpress any (QED!) $\mathcal{O}\left(\alpha^5\right)$ result in terms of the running $\alpha^{\overline{MS}}$ or vice versa
- Asymptotic contributions to the VP part of $(g-2)_\mu$ in order α^5 are computed and support the numerical result of the group of Kinoshita