# LoopFest VIII

*Radiative Corrections for the LHC and ILC*

## *D*-Dimensional Generalized Unitarity
## and
## Top Quarks

**Markus Schulze**

in collaboration with K. Melnikov

JOHNS HOPKINS UNIVERSITY

The traditional way to do a QCD 1-loop calculation:

- generate all Feynman diagrams
- reduction to scalar integrals
- reduction to spin and color structures
- numerical evaluation

This talk: ***D*-Dimensional Generalized Unitarity**

- completely orthogonal approach
- basic ingredients are tree level amplitudes
- better scaling with increasing number of external legs
- method is ready for phenomenology

application: **Top Quark Pair Production**

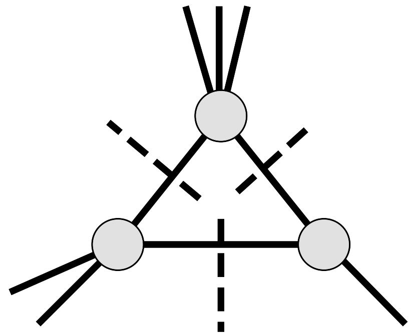
# *D*-Dimensional Generalized Unitarity

[van Neerven]
[Bern,Dixon,Dunbar,Kosower,Morgan]
[Giele,Kunszt,Melnikov]

cuts in $D\neq 4$ dimensions

[Britto,Cachazo,Feng]

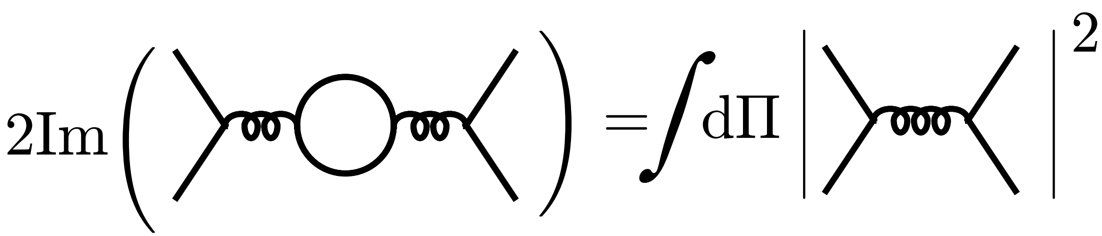
cut more than two propagators



**$D$-Dimensional Generalized Unitarity**

[Bern,Dixon,Dunbar,Kosower]

optical theorem



## Master Integral Basis

$$A^{\text{1-loop}} = \sum_j c_j I_j$$

$$I = \int \frac{d^D \ell}{2\pi} \frac{1}{D_0 \ldots D_N}$$

## Unitarity

$$\text{Im}\left(A^{\text{1-loop}}\right) \sim \int d\Pi \, |A^{\text{tree}}|^2$$

$$\sum_j c_j \, \text{Im}\left(I_j\right) \sim \int d\Pi \, |A^{\text{tree}}|^2$$

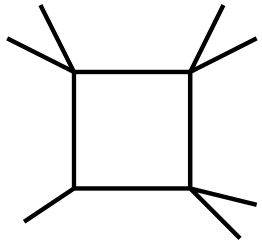
we know how to get the coefficients $c_j$ by calculating tree amplitudes important ingredient: **OPP Algorithm** [Ossola,Papadopoulos,Pittau]

- tensor integral reduction at the integrand level

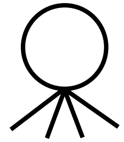
$$A^{\text{1-loop}} = \sum_j c_j I_j = \int \mathrm{d}^D\ell \, \frac{\text{Num}(\ell, \{p_i\})}{D_0 \ldots D_N} \qquad D_k = (\ell + p_k)^2 - m_k^2$$

- meshes very well with the cut-based unitarity method important ingredient: **OPP Algorithm** [Ossola,Papadopoulos,Pittau]

**Basic idea**:



...



**1.** partial fractioning: $\frac{\mathrm{Num}(\ell,\{p_i\})}{D_0 \ldots D_N} \sim \frac{\bar{d}(ijkl)}{D_i D_j D_k D_l} + \ldots + \frac{\bar{a}(i)}{D_i}$

**2.** choose vector basis that spans *physical space* + *transverse space*,
parametrize the coefficients:

$n_r \cdot p = 0$

$$\bar{c} = c \; + \; \tilde{c}_r \times \underbrace{\left(n_1^{\mu_1} \ldots n_r^{\mu_r} \ell_{\mu_1} \ldots \ell_{\mu_r}\right)}$$

$c \rightarrow \sum_j c_j I_j$

$\rightarrow$ vanishes after loop integration

**3.** we can extract the coefficients by considering only those loop momenta for which certain sets of inverse propagators vanish

*i.e.* $D_0 = D_1 = ... = D_k = 0$ *solve for* $\ell^\mu$

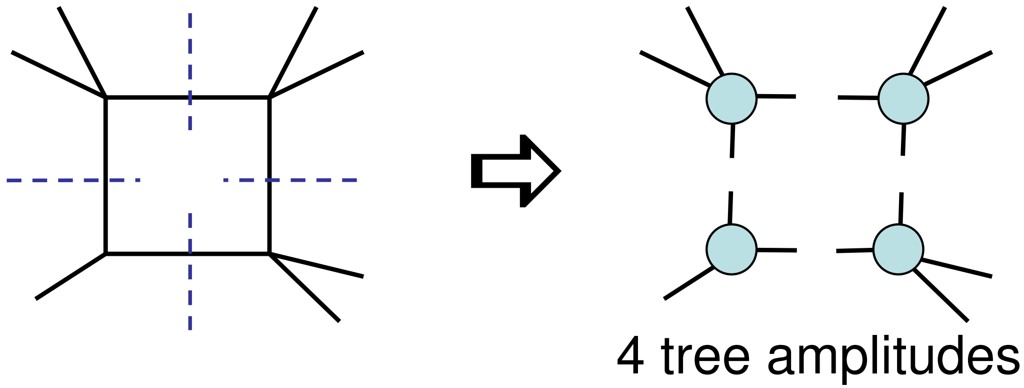
⇨ virtual particles go on-shell

**3.** we can extract the coefficients by considering only those loop momenta for which certain sets of inverse propagators vanish

*i.e.* $D_0 = D_1 = ... = D_k = 0$ *solve for* $\ell^\mu$

⇨ virtual particles go on-shell

*this is where OPP + unitarity mesh*




$$\mathrm{Num}(\ell, \{p_i\}) \to \Pi_k A_k^{\mathrm{tree}}(\ell, \{p_i\})$$

⇨ **basic ingredients are on-shell tree level amplitudes**

(with complex kinematics)

*rational terms*

*the full truth:*

$$A^{\text{1-loop}} = \sum_j c_j I_j + \mathcal{R}$$

*rational terms*: originate from $\varepsilon$ -dependent terms in coefficients

⇨ need to consider *D*-dependent expressions (*D*≠4) *rational terms*

*the full truth:*

$$A^{\text{1-loop}} = \sum_j c_j I_j + \mathcal{R}$$

*rational terms*: originate from $\varepsilon$ -dependent terms in coefficients

⇨ need to consider *D*-dependent expressions (*D*≠4)

Using our knowledge about the particular structure of *D*-dependence of $A^{\text{1-loop}}$, we can construct two *copies of QCD* in *D*=6 and *D*=8 to obtain the full result for *D* around 4. [Giele,Kunszt,Melnikov]

⇨ allows for straight-forward numerical implementation using helicity amplitudes

⇨ all we need are:

- spinors, gamma matrices and polarization vectors in *D*=6,8 dimensions
- loop momenta restricted to 5-dimensional subspace
- external momenta restricted to 4-dimensional subspace

# What has been achieved so far?

D-Dim. Generalized Unitarity ↘

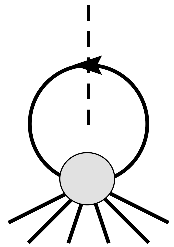
- multi-gluon amplitudes: *Rocket:* [Giele, Zanderighi], ✗
  [Lazopoulos], ✗
  [Giele, Winter] ✗
- $\mathrm{t}\,\bar{\mathrm{t}} + 3g$ amplitudes: [Ellis, Giele, Kunszt, Melnikov] ✗
- *CutTools* + *HELAS*: [van Hameren,Papadopoulos,Pittau]
  results for 10 amplitudes from 2007 Les Houches wishlist
- W+3jet **cross section**:

  *Rocket / MCFM:* [Ellis, Giele, Kunszt, Melnikov, Zanderighi] ✗

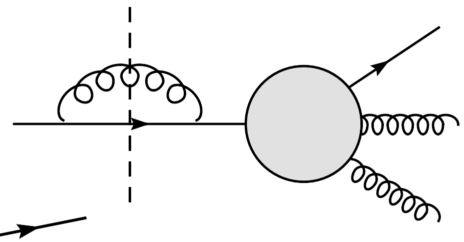
  *Blackhat + SHERPA:* [Berger, Bern, Dixon, Cordero, Forde, Ita, Kosower, Maitre, Gleisberg]

# What's special about
# unitarity + massive particles ?

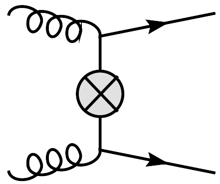
- single cuts (tadpole integrals)



- unitarity cuts on massive external lines



- mass counter term insertions

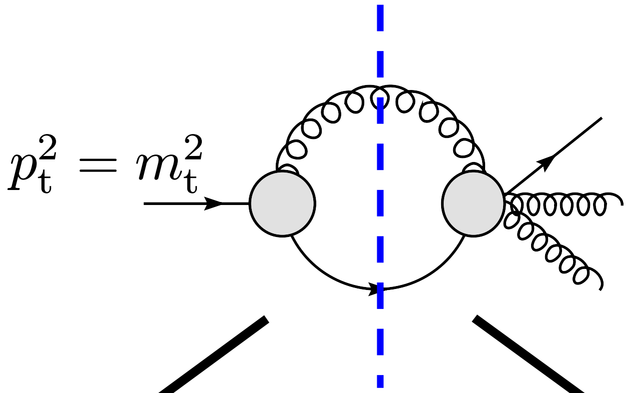


- potential instabilities close to threshold

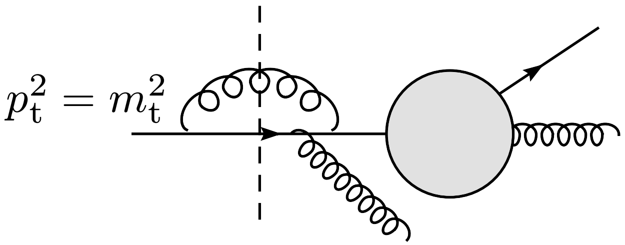
# Unitarity and self-energy corrections on massive quarks lines:

**( general issue for unitarity based methods! )**

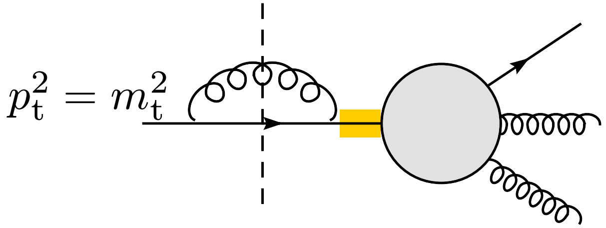
double cut:




regular contribution

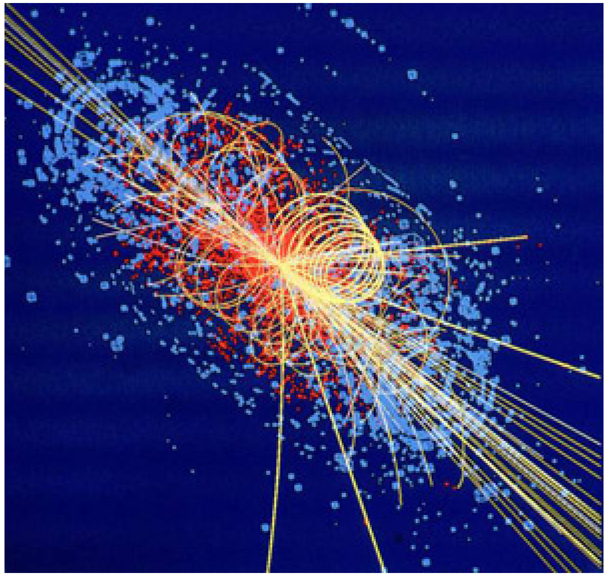



external self-energy contribution!



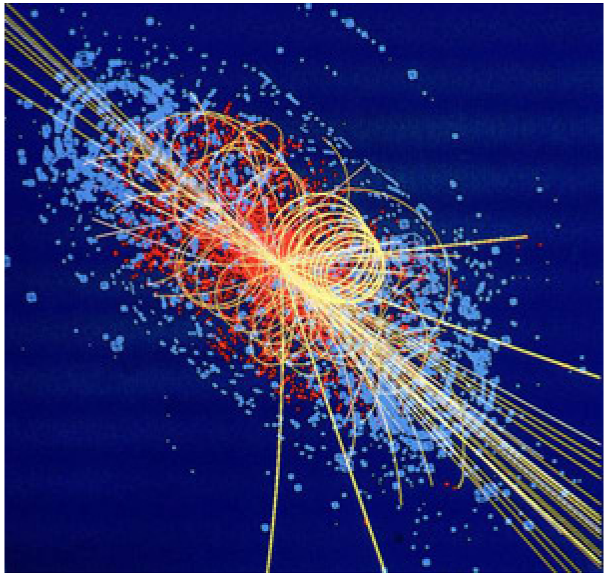

leads to on-shell propagator in tree amplitude

solution:
- discard terms that lead to those cuts (truncate BG recurrence relations)
- add wave function renormalization constants $\delta Z_t$ later



**This talk**:

**Cross Section for Top Quark Pair Production at Tevatron and LHC**



**This talk**:

**Cross Section for Top Quark Pair Production at Tevatron and LHC**

a lot of theoretical work accomplished:

[Ellis,Sexton] [Beenakker,Kuijf,van Neerven,Smith]
[Nason,Dawson,Ellis] [Meng,Schuler] — NLO results for $\bar{t}t$

[Bernreuther,Brandenburg,Si,Uwer] — NLO + decay with spin correlations

[Beenakker,Denner,Hollik,Mertig,Sack,Wackeroth] [Kühn,Scharf,Uwer] [Bernreuther,Fücker,Si] — electroweak corrections

[Dittmaier,Weinzierl,Uwer] — NLO results for $\bar{t}t + \mathrm{jet}$

+ MCFM [Campbell,Ellis]
+ soft gluon resummation, threshold logs...
+ many other phenomenological studies...

⇨ **good test ground for new methods**

# Implementation:

- *Rocket*-like `Fortran90` program:

$$0 \to t\bar{t} + N \text{ gluons}$$

$$0 \to t\bar{t} + q\bar{q} + N \text{ gluons}$$

at NLO QCD

(including $N_f$-terms with massive top quark loop)

- fully numerical implementation:

  helicity amplitudes via Berends-Giele recurrence relations

- all one needs for: $\text{pp} \to t\bar{t} + X$ ,

  $\text{pp} \to t\bar{t} + \text{jet} + X$

| | $\mathrm{pp} \to \mathrm{t\bar{t}} + X$ | | $\mathrm{pp} \to \mathrm{t\bar{t}} + \mathrm{jet} + X$ | |
|---|---|---|---|---|
| partonic channels: | LO + 1-loop: $gg, q\bar{q}$ | | LO + 1-loop: $gg, q\bar{q}, gq, g\bar{q}$ | |
| 1-loop topologies: | $0 \to \mathrm{t\bar{t}}gg$ | $0 \to \mathrm{t\bar{t}}q\bar{q}$ | $0 \to \mathrm{t\bar{t}}ggg$ | $0 \to \mathrm{t\bar{t}}q\bar{q}g$ |
| diagrams: | *31* | *10* | *354* | *94* |
| primitive ampl.: | *8* | *5* | *36* | *18* |
| evaluation time /prim.ampl./helicity [Intel Xeon 2.8GHz] | *3-5* msec | | *10-40* msec | |

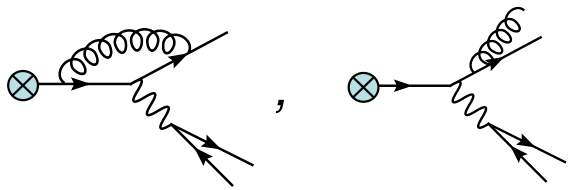
| | $\mathrm{pp} \to \mathrm{t\bar{t}} + X$ | | $\mathrm{pp} \to \mathrm{t\bar{t}} + \mathrm{jet} + X$ | |
|---|---|---|---|---|
| partonic channels: | LO + 1-loop: $gg, q\bar{q}$ | | LO + 1-loop: $gg, q\bar{q}, gq, g\bar{q}$ | |
| 1-loop topologies: | $0 \to \mathrm{t\bar{t}}gg$ | $0 \to \mathrm{t\bar{t}}q\bar{q}$ | $0 \to \mathrm{t\bar{t}}ggg$ | $0 \to \mathrm{t\bar{t}}q\bar{q}g$ |
| diagrams: | *31* | *10* | *354* | *94* |
| primitive ampl.: | *8* | *5* | *36* | *18* |
| evaluation time /prim.ampl./helicity `[Intel Xeon 2.8GHz]` | *3-5* msec | | *10-40* msec | |

- helicity formalism allows us to implement decay matrix elements

$$\mathrm{t\bar{t}} \xrightarrow{\text{NWA}} \mathrm{b\bar{b}}\, l^+\, l^-\, \nu\, \bar{\nu}$$

*including spin correlations*

*at NLO:*



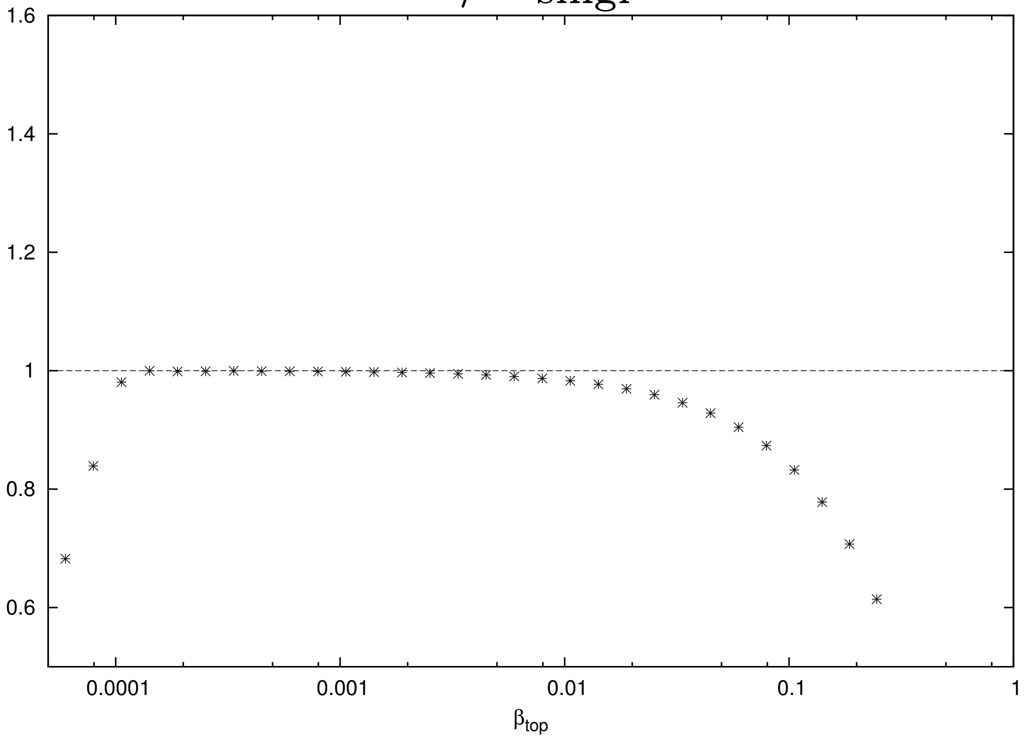
- caching of BG-currents
- control over numerical stability: switch to quadruple precision if necessary

# Numerical stability:

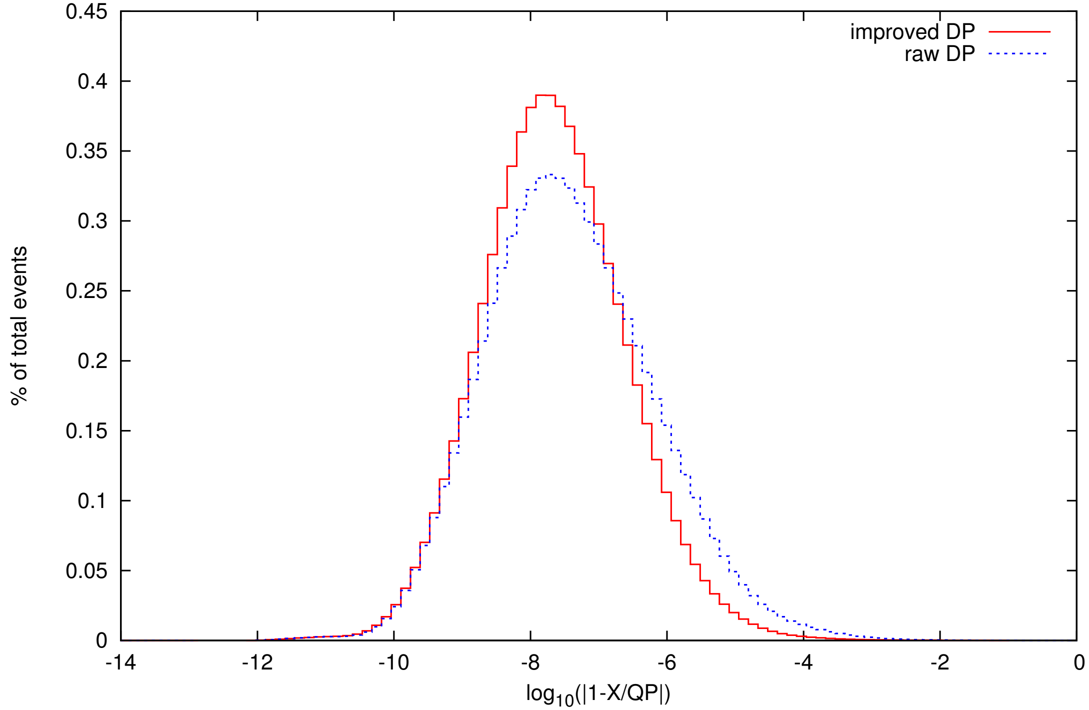
$t\bar{t}$:
- no stability issues
- checked threshold effects

$t\bar{t} + \text{jet}$:
- switch to quadruple precision if necessary

$A^{\text{1-loop}} / A^{\text{soft}}_{\text{singl}}$




$A^{\text{finite}}_{\text{L}}(1_{\bar{t}}, 4_g, 3_g, 5_g, 2_t)$




$$A^{\text{soft}}_{\text{singl}} = \frac{\alpha_s \pi}{2} A^{\text{tree}} \frac{C_F}{\beta}, \qquad \beta = \sqrt{1 - 4m^2_{\text{top}}/\hat{s}}$$

# Checks:

☑ checked coefficients of singular part ($1/\varepsilon^2, 1/\varepsilon$ ) for every primitive amplitude

☑ cancellation of $\varepsilon$ -poles

☑ check of gauge invariance ($\epsilon_\mu \tilde{\mathcal{M}}^\mu \to p_\mu \tilde{\mathcal{M}}^\mu$ )

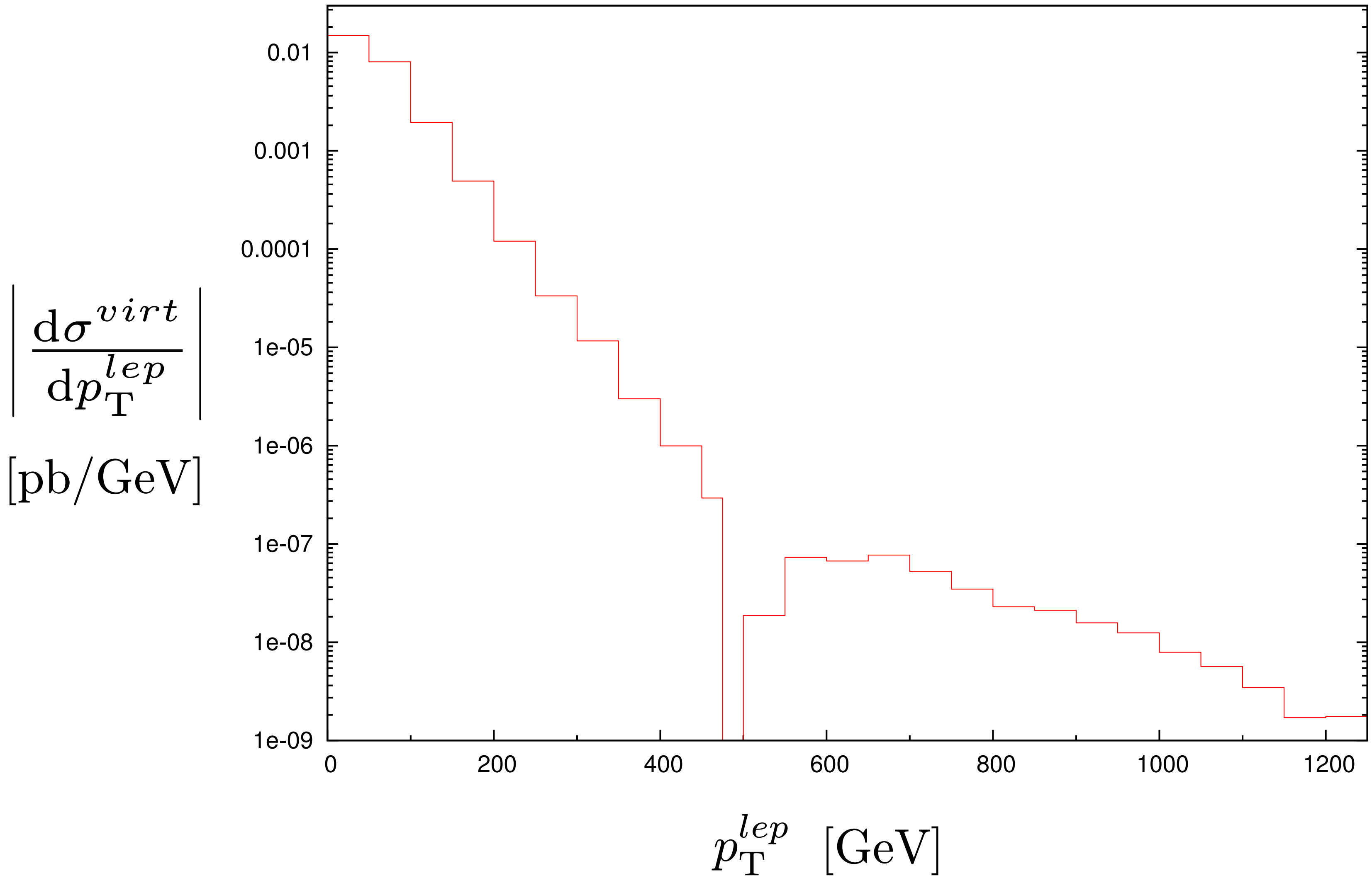
$\mathrm{t\bar{t}}:$ ☑ virtual corrections checked against MCFM

$\mathrm{t\bar{t} + jet}:$ ☐ check against [Dittmaier,Uwer,Weinzierl], partially checked

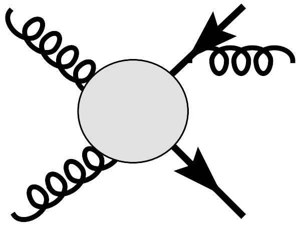
**Preliminary**

contribution of virtual correction to cross section at LHC for

$$\mathrm{pp} \to \mathrm{t\bar{t}} \to \mathrm{b\bar{b}} l^+ l^- \nu \bar{\nu}$$



## Real corrections:



- dipole subtraction formalism [Catani,Dittmaier,Seymour,Trocsanyi]

  semi-automated generation: Mathematica→FORM→Fortran95

$\mathrm{t\bar{t}} + X:$

- ☑ all matrix elements are checked against MadGraph
- ☑ all dipoles implemented and checked
- ☐ all integrated dipoles implemented, debugging phase

$\mathrm{t\bar{t}} + \mathrm{jet} + X:$

- ☐ under construction

# Summary:

## *D*-dimensional generalized unitarity...

... is a robust and transparent method to calculate 1-loop corrections

... basic ingredients are on-shell tree amplitudes

... is ready for phenomenology

application to: **Top quark pair production**

aiming at full NLO cross section for

$$\mathrm{pp} \rightarrow \mathrm{t}\bar{\mathrm{t}} + X$$

$$\xrightarrow{\text{NWA}} \mathrm{b}\bar{\mathrm{b}}\, l^+\, l^-\, \nu\, \bar{\nu}$$

$$\mathrm{pp} \rightarrow \mathrm{t}\bar{\mathrm{t}} + \mathrm{jet} + X$$

# Extras

**Giele, Kunszt, Melnikov**
**JHEP 0804:049,2008.**
**arXiv:0801.2237** [hep-ph]

dimensional space time, the number of spin eigenstates changes. For example, massless spin-one particles in $D_s$ dimensions have $D_s - 2$ spin eigenstates while spinors in $D_s$ dimensions have $2^{(D_s-2)/2}$ spin eigenstates. In the latter case, $D_s$ should be even.

The spin density matrix for a massless spin-one particle with momentum $l$ and polarization vectors $e_\mu^{(i)}$ is given by

$$\sum_{i=1}^{D_s-2} e_\mu^{(i)}(l) e_\nu^{(i)}(l) = -g_{\mu\nu}^{(D_s)} + \frac{l_\mu b_\nu + b_\mu l_\nu}{l \cdot b}, \tag{5}$$

where $b_\mu$ is an arbitrary light-cone gauge vector associated with a particular choice of polarization vectors. Similarly, the spin density matrix for a fermion with momentum $l$ and mass $m$ is given by

$$\sum_{i=1}^{2^{(D_s-2)/2}} u^{(i)}(l)\overline{u}^{(i)}(l) = \not{l} + m = \sum_{\mu=1}^{D} l_\mu \gamma^\mu + m\ . \tag{6}$$

While, as we see from these examples, the number of spin eigenstates depends explicitly on the space-time dimensionality, the loop-momentum $l$ itself has implicit $D$-dependence. We can define the loop momentum as a $D$-dimensional vector, with the requirement $D \leq D_s$ [35]. We now extend the notion of dimensional dependence of the one-loop scattering amplitude in Eq. (1) by taking the sources of all unobserved particles in $D_s$-dimensional space-time

$$\mathcal{A}_{(D,D_s)}(\{p_i\},\{J_i\}) = \int \frac{d^D l}{i(\pi)^{D/2}} \frac{\mathcal{N}^{(D_s)}(\{p_i\},\{J_i\};l)}{d_1 d_2 \cdots d_N}. \tag{7}$$

The numerator function $\mathcal{N}^{(D_s)}(\{p_i\},\{J_i\};l)$ depends explicitly on $D_s$ through the number of spin eigenstates of virtual particles. However, the dependence of the numerator function on the loop momentum dimensionality $D$ emerges in a peculiar way. Since external particles are kept in four dimensions, the dependence of the numerator function on $D-4$ components of the loop momentum $l$ appears only through its dependence on $l^2$. Specifically

$$l^2 = \bar{l}^2 - \tilde{l}^2 = l_1^2 - l_2^2 - l_3^2 - l_4^2 - \sum_{i=5}^{D} l_i^2\ , \tag{8}$$

where $\bar{l}$ and $\tilde{l}$ denote four- and $(D-4)$-dimensional components of the vector $l$. It is apparent from Eq. (8) that there is no preferred direction in the $(D-4)$-dimensional subspace of the $D$-dimensional loop momentum space.

A simple, but important observation is that in one-loop calculations, the dependence of scattering amplitudes on $D_s$ is *linear*. This happens because, for such dependence to appear,

**Giele, Kunszt, Melnikov**
**JHEP 0804:049,2008.**
**arXiv:0801.2237** [hep-ph]

we need to have a closed loop of contracted metric tensors and/or Dirac matrices coming from vertices and propagators. Since only a single loop can appear in one-loop calculations, we find

$$\mathcal{N}^{(D_s)}(l) = \mathcal{N}_0(l) + (D_s - 4)\mathcal{N}_1(l). \qquad (9)$$

We emphasize that there is no explicit dependence on either $D_s$ or $D$ in functions $\mathcal{N}_{0,1}$.

For numerical calculations we need to separate the two functions $\mathcal{N}_{0,1}$. To do so, we compute the left hand side of Eq. (9) for $D_s = D_1$ and $D_s = D_2$ and, after taking appropriate linear combinations, obtain

$$\begin{aligned} \mathcal{N}_0(l) &= \frac{(D_2-4)\mathcal{N}^{(D_1)}(l) - (D_1-4)\mathcal{N}^{(D_2)}(l)}{D_2 - D_1}, \\ \mathcal{N}_1(l) &= \frac{\mathcal{N}^{(D_1)}(l) - \mathcal{N}^{(D_2)}(l)}{D_2 - D_1}. \end{aligned} \qquad (10)$$

Because both $D_1$ and $D_2$ are integers, amplitudes are numerically well-defined. We will comment more on possible choices of $D_{1,2}$ in the forthcoming sections; here suffice it to say that if fermions are present in the loop, we have to choose *even* $D_1$ and $D_2$.

Having established the $D_s$-dependence of the amplitude, we discuss analytic continuation for sources of unobserved particles. We can interpolate $D_s$ either to $D_s \to 4-2\epsilon$ (the t'Hooft-Veltman (HV) scheme) [34] or to $D_s \to 4$ (the four-dimensional helicity (FDH) scheme) [35]. The latter scheme is of particular interest in supersymmetric (SUSY) calculations since all SUSY Ward identities are preserved. We see from Eq. (9) that the difference between the two schemes is simply $-2\epsilon\mathcal{N}_1$.

We now substitute Eq.( 10) into Eq. (7). Upon doing so, we obtain explicit expressions for one-loop amplitudes in HV and FDH schemes. We derive

$$\begin{aligned} \mathcal{A}^{\rm FDH} &= \left(\frac{D_2-4}{D_2-D_1}\right)\mathcal{A}_{(D,D_s=D_1)} - \left(\frac{D_1-4}{D_2-D_1}\right)\mathcal{A}_{(D,D_s=D_2)}, \\ \mathcal{A}^{\rm HV} &= \mathcal{A}^{\rm FDH} - \left(\frac{2\epsilon}{D_2-D_1}\right)\left(\mathcal{A}_{(D,D_s=D_1)} - \mathcal{A}_{(D,D_s=D_2)}\right). \end{aligned} \qquad (11)$$

We emphasize that $D_s = D_{1,2}$ amplitudes on the r.h.s. of Eq. (11) are conventional one-loop scattering amplitudes whose numerator functions are computed in higher-dimensional space-time, i.e. all internal metric tensors and Dirac gamma matrices are in integer $D_s = D_{1,2}$ dimensions. The loop integration is in $D \leq D_s$ dimensions. It is important that explicit dependence on the regularization parameter $\epsilon = (4-D)/2$ is not present in these amplitudes.

**Giele, Kunszt, Melnikov**
**JHEP 0804:049,2008.**
**arXiv:0801.2237** [hep-ph]

function. Hence the integrand of the $N$-particle amplitude in Eq. (1) can be parameterized as

$$\frac{\mathcal{N}^{(D_s)}(l)}{d_1 d_2 \cdots d_N} = \sum_{[i_1|i_5]} \frac{\overline{e}^{(D_s)}_{i_1 i_2 i_3 i_4 i_5}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} + \sum_{[i_1|i_4]} \frac{\overline{d}^{(D_s)}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{[i_1|i_3]} \frac{\overline{c}^{(D_s)}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\overline{b}^{(D_s)}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\overline{a}^{(D_s)}_{i_1}(l)}{d_{i_1}}. \quad (13)$$

where the dependence on the external momenta and sources are suppressed. From four-dimensional unitarity we know that computation of each cut of the scattering amplitude is simplified if convenient parameterization of the residue is chosen. We now discuss how these parameterizations change when $D_s$-dimensional unitarity cuts are considered.

### A. Pentuple residue

To calculate the pentuple residue, we choose momentum $l$ such that five inverse propagators in Eq. (13) vanish. We define

$$\overline{e}^{(D_s)}_{ijkmn}(l_{ijkmn}) = \mathrm{Res}_{ijkmn}\left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1 \cdots d_N}\right). \quad (14)$$

The momentum $l_{ijkmn}$ satisfies the following set of equations $d_i(l_{ijkmn}) = \cdots = d_n(l_{ijkmn}) = 0$. The solution is given by

$$l^\mu_{ijkmn} = V_5^\mu + \sqrt{\frac{-V_5^2 + m_n^2}{\alpha_5^2 + \cdots + \alpha_D^2}}\left(\sum_{h=5}^{D} \alpha_h n_h^\mu\right), \quad (15)$$

where $m_n$ is the mass in the propagator $d_n$ which is chosen to be as $d_n = l^2 - m_n^2$ by adjusting the reference vector $q_0$. The parameters $\alpha_h$ can be chosen freely. The four-dimensional vector $V_5^\mu$ depends only on external momenta and propagator masses. It is explicitly constructed using the Vermaseren-van Neerven basis as outlined in Ref. [24]. The $D-4$ components of the vector $l_{ijkmn}$ are necessarily non-vanishing; for simplicity we may choose $l_{ijkmn}$ to be five-dimensional, independent of $D_s$. We will see below that this is sufficient to determine pentuple residue.

To restrict the functional form of the pentuple residue $\overline{e}_{ijkmn}(l)$ we apply the same reasoning as in four-dimensional unitarity case, supplemented with the requirement that $\overline{e}^{(D_s)}_{ijkmn}(l)$

**Giele, Kunszt, Melnikov**
**JHEP 0804:049,2008.**
**arXiv:0801.2237** *[hep-ph]*

depends only on even powers of $s_e$; this requirement is a necessary consequence of the discussion around Eq. (8). These considerations lead to the conclusion that the pentuple residue is independent of the loop momentum

$$\overline{e}^{(D_s)}_{ijkmn}(l) = e^{(D_s,(0))}_{ijkmn}. \qquad (16)$$

To calculate $e^{(0)}$ in the FDH scheme, we employ Eq. (10) and obtain

$$e^{(0),\mathrm{FDH}}_{ijkmn} = \left(\frac{D_2-4}{D_2-D_1}\right)\mathrm{Res}_{ijkmn}\left(\frac{\mathcal{N}^{(D_1)}(l)}{d_1\cdots d_N}\right) - \left(\frac{D_1-4}{D_2-D_1}\right)\mathrm{Res}_{ijkmn}\left(\frac{\mathcal{N}^{(D_2)}(l)}{d_1\cdots d_N}\right). \qquad (17)$$

The calculation of the residues of the amplitude on the r.h.s. of Eq. (17), is simplified by their factorization into products of tree amplitudes

$$\mathrm{Res}_{ijkmn}\left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1\cdots d_N}\right) = \sum \mathcal{M}(l_i;p_{i+1},\ldots,p_j,-l_j)\times\mathcal{M}(l_j;p_{j+1},\ldots,p_k;-l_k) \qquad (18)$$
$$\times\mathcal{M}(l_k;p_{k+1},\ldots,p_m;-l_m)\times\mathcal{M}(l_m;p_{m+1},\ldots,p_n;-l_n)\times\mathcal{M}(l_n;p_{n+1}...,p_i;-l_i).$$

Here, the summation is over all different quantum numbers of the cut lines. In particular, we have to sum over polarization vectors of the cut lines. This generates explicit $D_s$ dependence of the residue, as described in the previous section. Note that the complex momenta $l^\mu_h = l^\mu + q^\mu_h$ are on-shell due to the unitarity constraint $d_h = 0$.

### B. Quadrupole residue

The construction of the quadrupole residue follows the discussion of the previous subsection and generalizes the four-dimensional case studied in [23, 24]. We define

$$\overline{d}^{(D_s)}_{ijkn}(l) = \mathrm{Res}_{ijkn}\left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1\cdots d_N} - \sum_{[i_1|i_5]}\frac{e^{(D_s,(0))}_{i_1i_2i_3i_4i_5}}{d_{i_1}d_{i_2}d_{i_3}d_{i_4}d_{i_5}}\right), \qquad (19)$$

where the last term in the r.h.s. is the necessary subtraction of the pentuple cut contribution. We now specialize to the FDH scheme. In this case, the most general parameterization of the quadrupole cut is given by

$$\overline{d}^{\mathrm{FDH}}_{ijkn}(l) = d^{(0)}_{ijkn} + d^{(1)}_{ijkn}s_1 + (d^{(2)}_{ijkn} + d^{(3)}_{ijkn}s_1)s_e^2 + d^{(4)}_{ijkn}s_e^4, \qquad (20)$$

where $s_1 = l\cdot n_1$. We used the fact that, in renormalizable quantum field theories, the highest rank of a tensor integral that may contribute to a quadrupole residue is four and

**Giele, Kunszt, Melnikov**
**JHEP 0804:049,2008.**
**arXiv:0801.2237 [hep-ph]**

$$s_e^2 = (\tilde{l} \cdot n_5)^2$$

find

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{(D-2)(D-4)}{4} I^{D+4}_{i_1 i_2 i_3 i_4},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2 i_3},$$
$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2}} = -\frac{(D-4)}{2} I^{D+2}_{i_1 i_2}. \tag{26}$$

Using Eq. (26), we arrive at the following representation of the scattering amplitude

$$\mathcal{A}_{(D)} = \sum_{[i_1|i_5]} e^{(0)}_{i_1 i_2 i_3 i_4 i_5} \, I^{(D)}_{i_1 i_2 i_3 i_4 i_5}$$
$$+ \sum_{[i_1|i_4]} \left( d^{(0)}_{i_1 i_2 i_3 i_4} \, I^{(D)}_{i_1 i_2 i_3 i_4} - \frac{D-4}{2} d^{(2)}_{i_1 i_2 i_3 i_4} \, I^{(D+2)}_{i_1 i_2 i_3 i_4} + \frac{(D-4)(D-2)}{4} d^{(4)}_{i_1 i_2 i_3 i_4} \, I^{(D+4)}_{i_1 i_2 i_3 i_4} \right)$$
$$+ \sum_{[i_1|i_3]} \left( c^{(0)}_{i_1 i_2 i_3} \, I^{(D)}_{i_1 i_2 i_3} - \frac{D-4}{2} c^{(9)}_{i_1 i_2 i_3} \, I^{(D+2)}_{i_1 i_2 i_3} \right)$$
$$+ \sum_{[i_1|i_2]} \left( b^{(0)}_{i_1 i_2} \, I^{(D)}_{i_1 i_2} - \frac{D-4}{2} b^{(9)}_{i_1 i_2} \, I^{(D+2)}_{i_1 i_2} \right) + \sum_{i_1=1}^{N} a^{(0)}_{i_1} \, I^{(D)}_{i_1} \,. \tag{27}$$

We emphasize that the explicit $D$-dependence on the r.h.s. of Eq. (27) is the consequence of our choice of the basis for master integrals in Eq. (26).

We note that the above decomposition is valid for any value of $D$. We can now interpolate the loop integration dimension $D$ to $D \to 4 - 2\epsilon$. The extended basis of master integrals that we employ provides a clear separation between cut-constructible and rational parts of the amplitude. The cut-constructible part is given by the integrals in $D$-dimensions in Eq. (27), while the rational part is given by the integrals in $D+2$ and $D+4$ dimensions. However, it is possible to use smaller basis of master integrals by rewriting integrals $\{I^{(D+4)}_{i_1 i_2 i_3 i_4}, I^{(D+2)}_{i_1 i_2 i_3 i_4}, I^{(D+2)}_{i_1 i_2 i_3}, I^{(D+2)}_{i_1 i_2}\}$ in terms of $\{I^{(D)}_{i_1 i_2 i_3 i_4}, I^{(D)}_{i_1 i_2 i_3}, I^{(D)}_{i_1 i_2}\}$ using the integration-by-parts techniques.

Since we are interested in NLO computations, we only need to consider the limit $\epsilon \to 0$ in Eq. (27) and neglect contributions of order $\epsilon$. This leads to certain simplifications. First, in this limit, we can re-write the scalar 5-point master integral as a linear combination of four-point master integrals up to $\mathcal{O}(\epsilon)$ terms. If we employ this fact in Eq. (27), we obtain

$$\lim_{D \to 4-2\epsilon} \left( \sum_{[i_1|i_5]} e^{(0)}_{i_1 \cdots i_5} \, I^{(D)}_{i_1 \cdots i_5} + \sum_{[i_1|i_4]} d^{(0)}_{i_1 \cdots i_4} \, I^{(D)}_{i_1 \cdots i_4} \right) = \sum_{[i_1|i_4]} \tilde{d}^{(0)}_{i_1 \cdots i_4} \, I^{(4-2\epsilon)}_{i_1 \cdots i_4} + \mathcal{O}(\epsilon) \,. \tag{28}$$