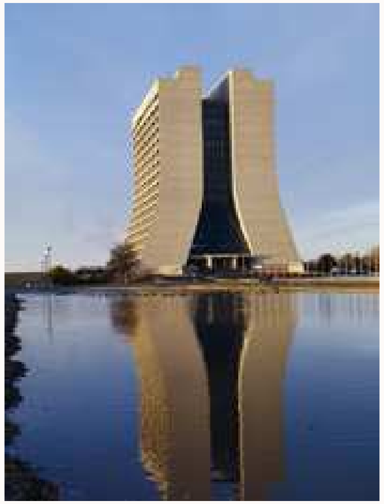
Numerical calculation of one-loop gluon amplitudes

[LoopFest8 @ Madison]

Jan Winter [a]

– Fermilab –



- *From tree-level to one-loop amplitudes – EGKM algorithm*
- *Results for colour-ordered amplitudes*
- *Outline of the method for colour-dressed amplitudes – **work in progress***

[a] In collaboration with: W. Giele and Z. Kunszt

NLO calculations

- Feynman diagram calculations: computational algorithms of at least factorial complexity
- bottleneck: virtual corrections (tensor-integral reductions generate large # of terms)
- @ tree level: algorithms of polynomial or, incl. colour, exponential complexity exist ($\tau \sim N^{\#}$ or $\#^{N}$)
  recursive methods efficiently re-use recurring groups of offshell Feynman graphs
- @ loop level: generalized unitarity-cut methods factorize one-loop into tree amplitudes
  computing time grows with # of cuts & depends on algorithm employed at tree level

Goal ⇒ ***provide algorithm(s) [tools] of exponential complexity to calculate virtual corrections***

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⇒ ***Generalized unitarity methods – active field of research***

- Britto et al.
- Bern et al. – BlackHat project and code.
- Hameren, Ossola et al. – Helac, CutTools code.
- Ellis et al. – "Rocket Science".
- Lazopoulos – N gluon code.

Decomposing one-loop amplitudes

➡ ***into a linear sum of scalar box, triangle, bubble and tadpole master integrals (cut-constructible part) and rational terms***

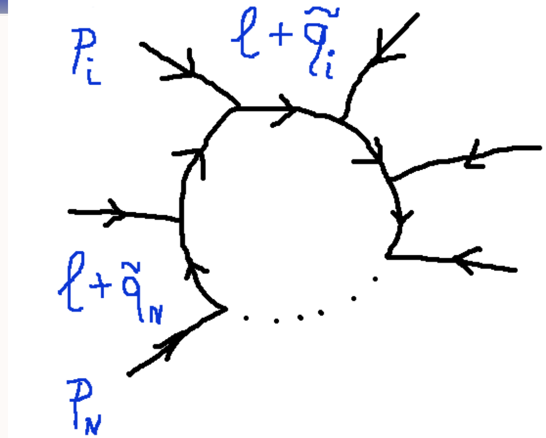
$$\mathcal{A}_N(\{p_i\}) = \sum_{[i_1|i_4]} d_{i_1 i_2 i_3 i_4} I^{(D)}_{i_1 i_2 i_3 i_4} + \sum_{[i_1|i_3]} c_{i_1 i_2 i_3} I^{(D)}_{i_1 i_2 i_3} + \sum_{[i_1|i_2]} b_{i_1 i_2} I^{(D)}_{i_1 i_2} + \sum_{[i_1|i_1]} a_{i_1} I^{(D)}_{i_1} + \mathcal{R}_N$$

- master integrals known in literature
- and implemented in various codes, e.g. QCDLoop [Ellis, Zanderighi] (QCDLoop.fnal.gov)
- to do: determination of the master-integral coefficients ⬅ unitarity techniques
- problem: extraction of lower-point coefficients
  "subtracting terms that are already included in higher-point contributions"
- solution: identify subtraction terms at the integrand level [Ossola, Papadopoulos, Pittau]
  partial fractioning of the integrand: expand over 4,3,2,1 propagator terms
  residues of pole terms contain master-integral coefficient plus finite number of spurious terms
  spurious terms vanish upon integration

note that $[i_1, i_M] = 1 \leq i_1 < i_2 < \ldots < i_M \leq N$ and $I^{(D)}_{i_1 \ldots i_M} = \int d^D\ell \frac{1}{d_{i_1} \ldots d_{i_M}}$

Ellis–Giele–Kunszt–Melnikov method.

➡ re-expressing the integrand $\qquad d_i(\ell) = (\ell + \tilde{q}_i)^2 - m_i^2$



$$\mathcal{A}_N^{(D_s)}(\{p_i\},\ell) \;=\; \frac{\mathcal{N}_0(\{p_i\},\ell) + (D_s - 4)\,\mathcal{N}_1(\{p_i\},\ell)}{d_1 d_2 \dots d_N} \;=$$

$$\sum_{[i_1|i_5]} \frac{\bar{e}^{(D_s)}_{i_1 i_2 i_3 i_4 i_5}(\ell)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} + \sum_{[i_1|i_4]} \frac{\bar{d}^{(D_s)}_{i_1 i_2 i_3 i_4}(\ell)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{[i_1|i_3]} \frac{\bar{c}^{(D_s)}_{i_1 i_2 i_3}(\ell)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\bar{b}^{(D_s)}_{i_1 i_2}(\ell)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\bar{a}^{(D_s)}_{i_1}(\ell)}{d_{i_1}}$$

- solving for numerator factors ➡ "the Left-Hand-Side"

$$\bar{e}^{(D_s)}_{i_1\cdots i_5}(\ell) = \mathrm{Res}_{i_1\cdots i_5}(\mathcal{A}_N^{(D_s)}(\ell))\,, \qquad \bar{d}^{(D_s)}_{i_1\cdots i_4}(\ell) = \mathrm{Res}_{i_1\cdots i_4}\left[\mathcal{A}_N^{(D_s)}(\ell) - \sum_{[j_1|j_5]} \frac{\bar{e}^{(D_s)}_{j_1 j_2 j_3 j_4 j_5}(\ell)}{d_{j_1} d_{j_2} d_{j_3} d_{j_4} d_{j_5}}\right]\,, \;\dots$$

need to find $D \le D_s$ dim $\quad \ell = \ell_{i_1 \dots i_M}$ $\quad$ such that $\quad d_j(\ell_{i_1 \dots i_M}) = 0$ $\quad$ for $\quad j = i_1, \dots, i_M$

define $\quad \mathrm{Res}_{i_1 \dots i_M}(\mathcal{A}_N^{(D_s)}(\ell)) = \{d_{i_1}(\ell) \dots d_{i_M}(\ell) \times \mathcal{A}_N^{(D_s)}(\ell)\}|_{\ell = \ell_{i_1 \dots i_M}}$

- find parametric form of residues, removing spurious terms ➡ "the Right-Hand-Side"

box coefficient: $\quad \bar{d}^{(D_s)}_{i_1\cdots i_4}(\ell) \;=\; d^{(0)}_{i_1\cdots i_4} + \alpha_4\, d^{(1)}_{i_1\cdots i_4} + s_e^2\,[d^{(2)}_{i_1\cdots i_4} + \alpha_4\, d^{(3)}_{i_1\cdots i_4}] + s_e^4\, d^{(4)}_{i_1\cdots i_4}$

$$\Rightarrow \quad \int d^D\ell\, \frac{\bar{d}^{(D_s)}_{i_1\cdots i_4}(\ell)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} \;=\; d^{(0)}_{i_1\cdots i_4}\; I_{i_1\cdots i_4} \;-\; d^{(4)}_{i_1\cdots i_4}/6$$

Generating the Left-Hand-Side

- What is $\mathrm{Res}_{i_1\ldots i_M}(\mathcal{A}_N^{(D_s)}(\ell))$?

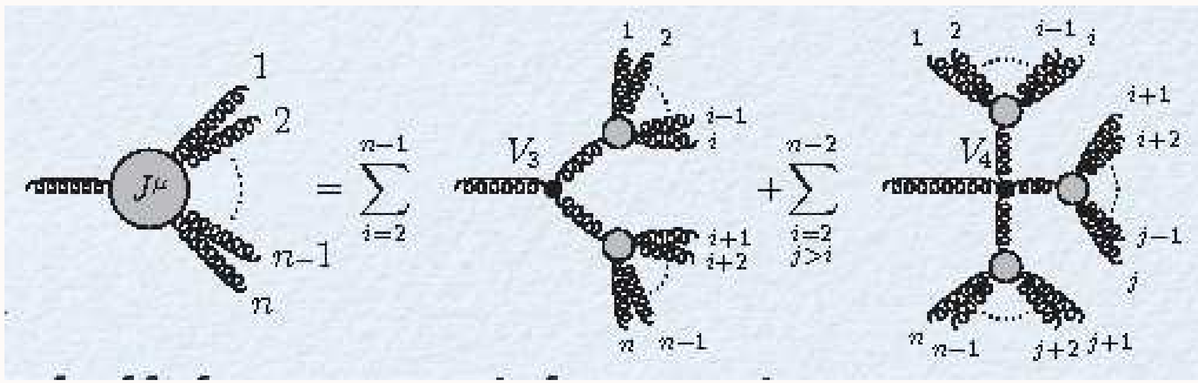
$$= \{\, d_{i_1}(\ell)\ldots d_{i_M}(\ell) \times \mathcal{A}_N^{(D_s)}(\ell)\,\}|_{d_{i_1}(\ell)=\cdots=d_{i_M}(\ell)=0}$$

- requires calculation of factorized un-integrated one-loop amplitude
- unitarity cuts: M on-shell propagators, amplitude factorizes into M tree-level amplitudes

$$\mathrm{Res}_{i_1\ldots i_M}(\mathcal{A}_N^{(D_s)}(\ell)) = \sum_{\{\lambda_1,\ldots,\lambda_M\}=1}^{D_s-2} \left(\prod_{k=1}^{M} \mathcal{M}^{(0)} \left(\ell_{i_k}^{(\lambda_k)}; p_{i_k+1},\ldots,p_{i_{k+1}}; -\ell_{i_{k+1}}^{(\lambda_{k+1})} \right) \right)$$

  - two D_s dimensional gluons with complex momenta and D_s-2 polarization states $(\ell_{i_k} = \ell + \tilde{q}_{i_k})$

- Berends–Giele recursion relations to calculate tree-level amplitudes
- very economical scheme
- <u>LHS:</u> take subtractions into account



C++ code

Implemented algorithm based on ...

[Ellis, Giele, Kunszt, arXiv:0708.2398] 4dim method, cut-constructible part

[Giele, Kunszt, Melnikov, arXiv:0801.2237] Ddim method, rational part

[Giele, Zanderighi, arXiv:0805.2152] application of Ddim method to pure gluons

- independent implementation (from scratch, no translation of Fortran routines)
- good xcheck of generalized-unitarity method and its results

N external gluons & their polarizations → _(leading-)colour-ordered 1-loop amplitude (FDH)_

- xchecks on numbers

  coefficients itself, poles (known analytically), final numbers (analytic and other calculations)

  gauge invariance, choice of ℓ, dimensionality (D and D_s variation)

- accuracy and numerical stability

$$\varepsilon_{\rm dp,sp} = \log_{10} \frac{|\mathcal{A}^{(1)({\rm dp,sp})}_{N,{\rm C++}} - \mathcal{A}^{(1)({\rm dp,sp})}_{N,{\rm anly}}|}{|\mathcal{A}^{(1)({\rm dp,sp})}_{N,{\rm anly}}|}\,, \qquad \varepsilon_{\rm fp} = \log_{10} \frac{2\,|\mathcal{A}^{(1)({\rm fp})}_{N,{\rm C++}}[1] - \mathcal{A}^{(1)({\rm fp})}_{N,{\rm C++}}[2]|}{|\mathcal{A}^{(1)({\rm fp})}_{N,{\rm C++}}[1]| + |\mathcal{A}^{(1)({\rm fp})}_{N,{\rm C++}}[2]|}$$

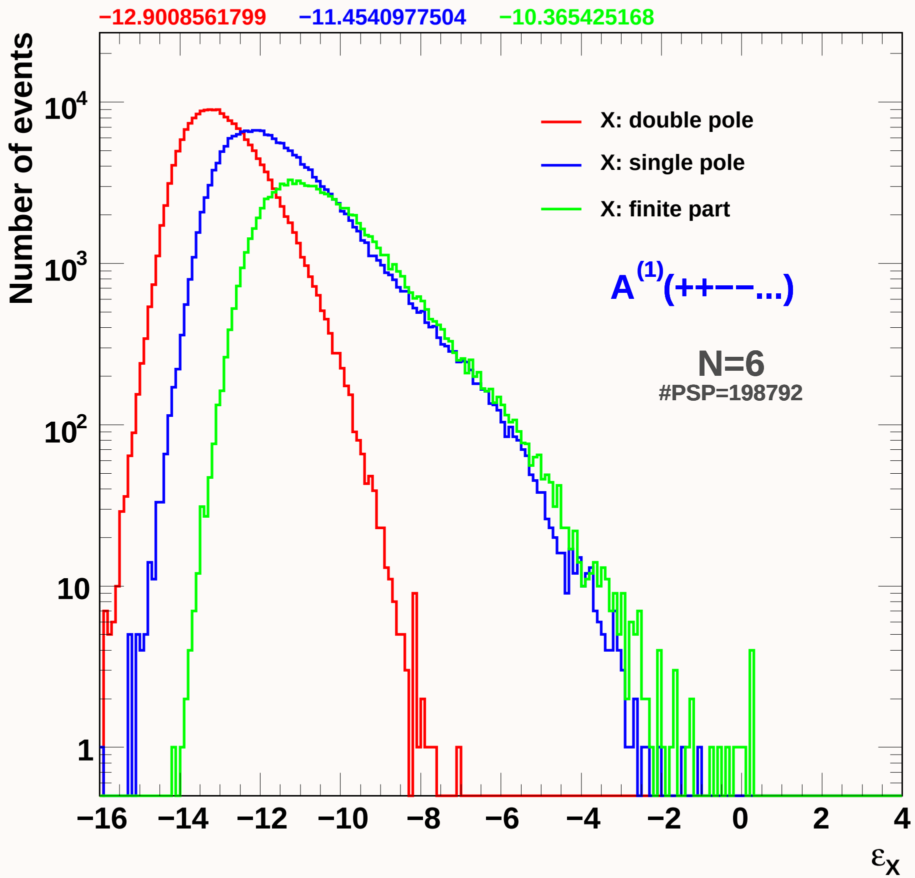
- efficiency – scaling of computing time with # of legs N $\rightarrow$ $\tau \sim N^9$

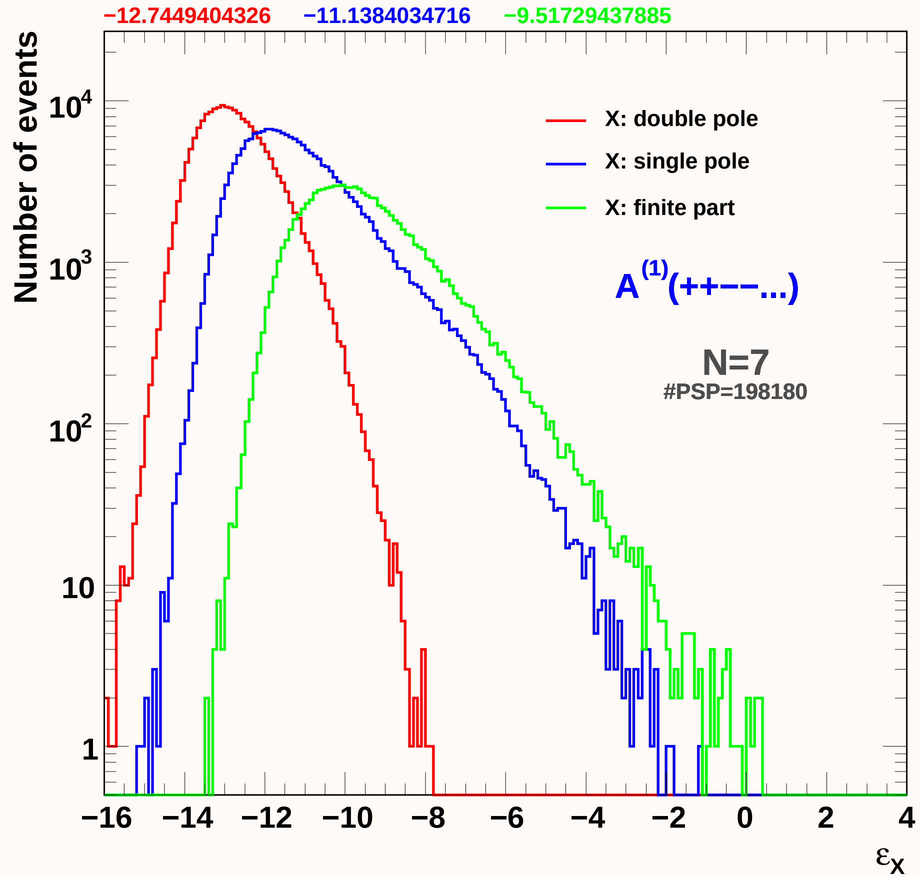
Accuracy

(preliminary) (all calculations in double precision only) [GIELE, WINTER, ARXIV:0902.0094]

- peak positions & tails are OK, comparable to Rocket (Rucola) results [GIELE, ZANDERIGHI]
- losing finite-part precision with $N = 10, 11$, lost for $N = 15$

  (double precision not enough, too many large numbers involved)





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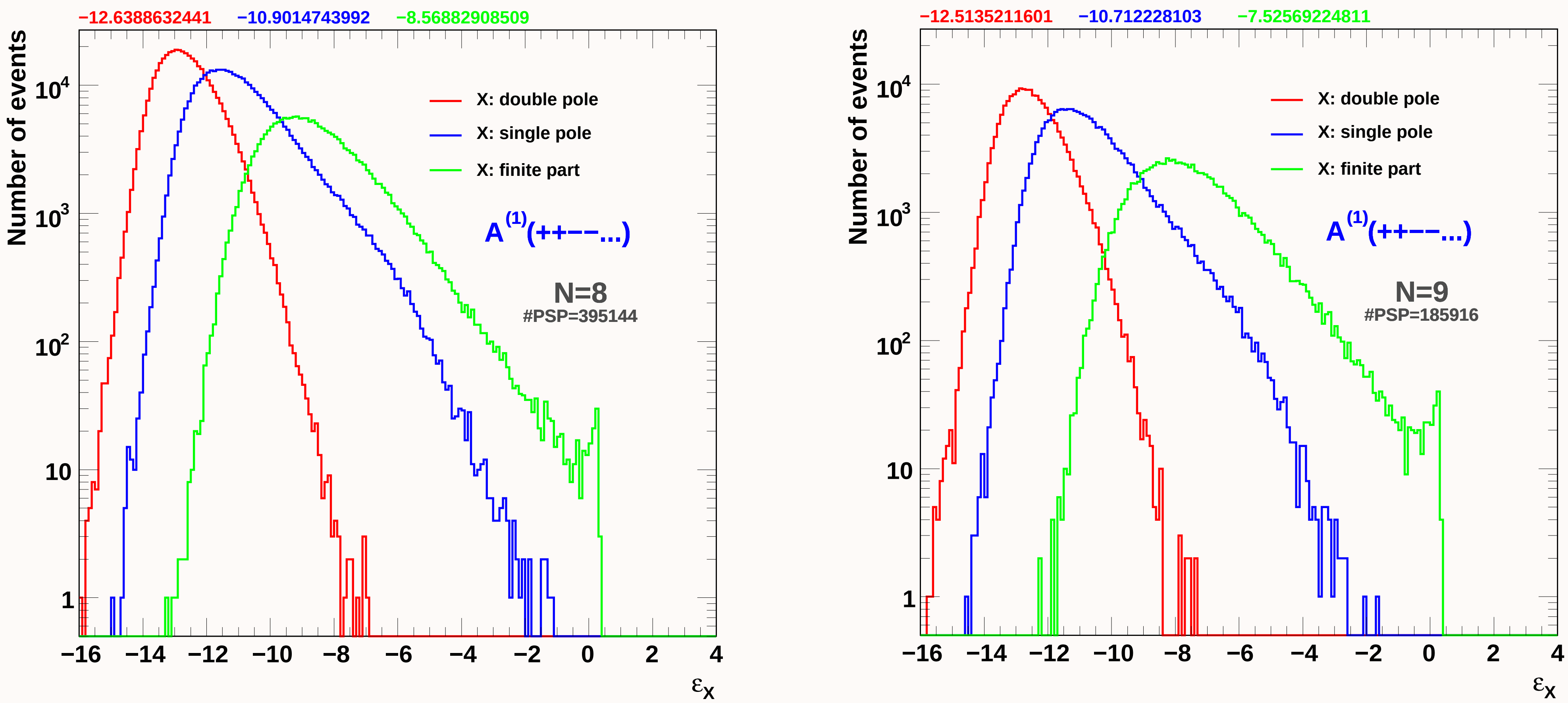

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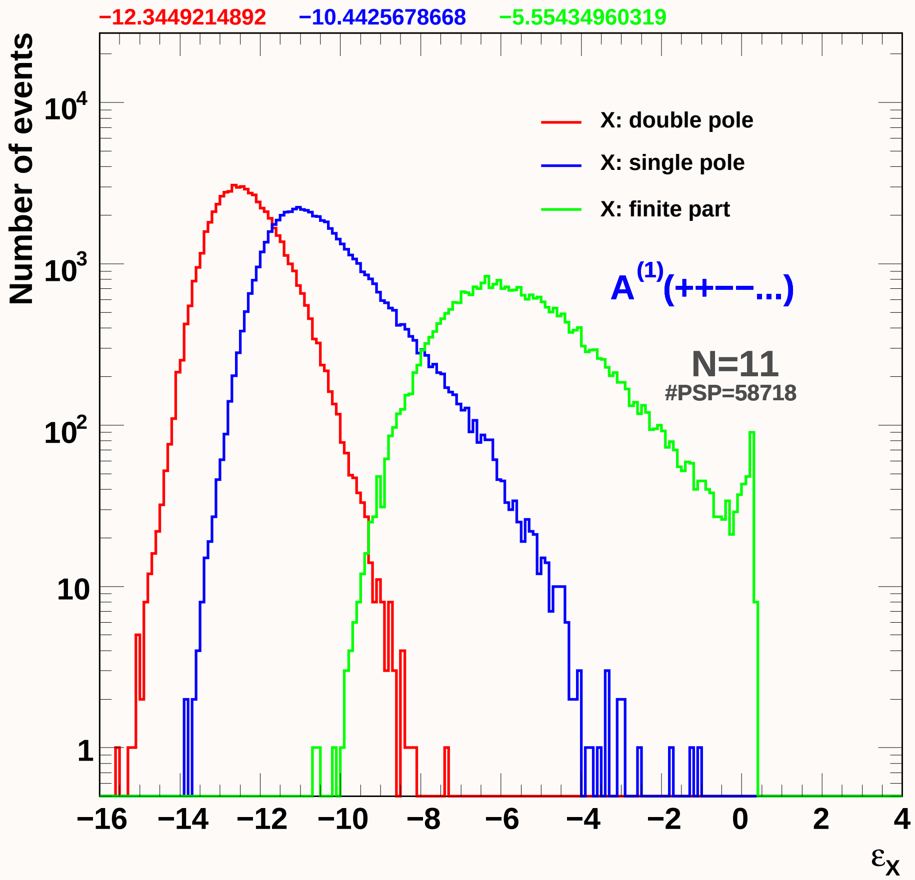

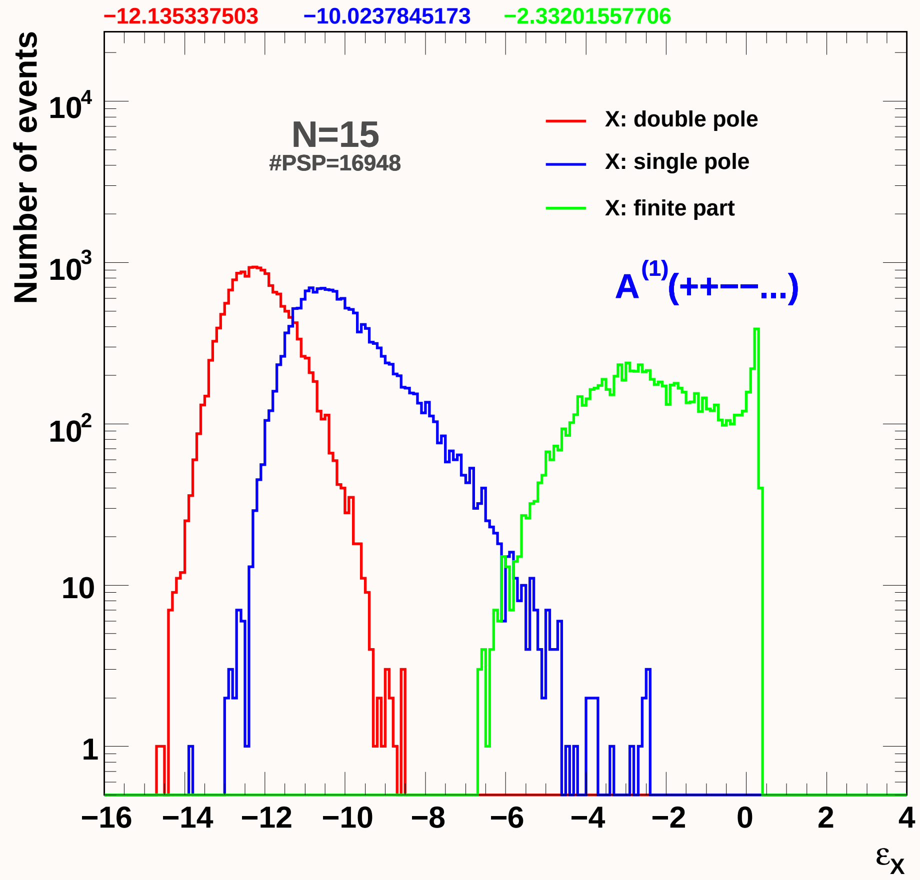

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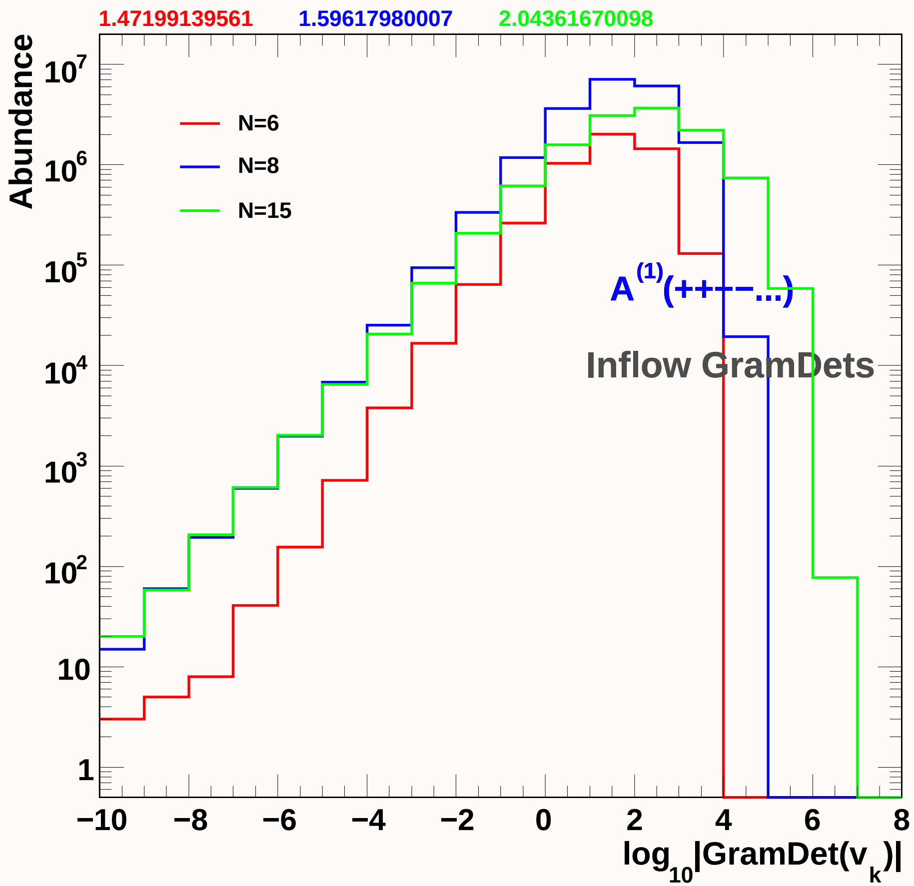
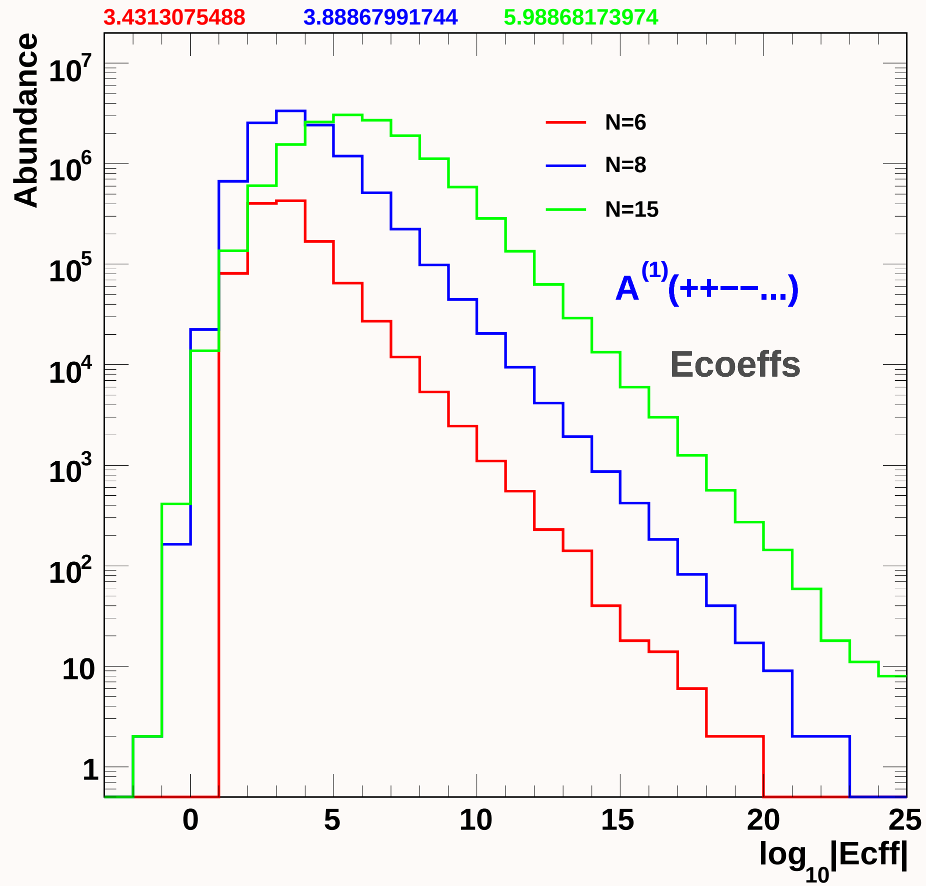
- range of numbers increases with N – Gram dets of external gluons and $e^{(0)}_{ijklm}$ coefficients may become small and large, respectively

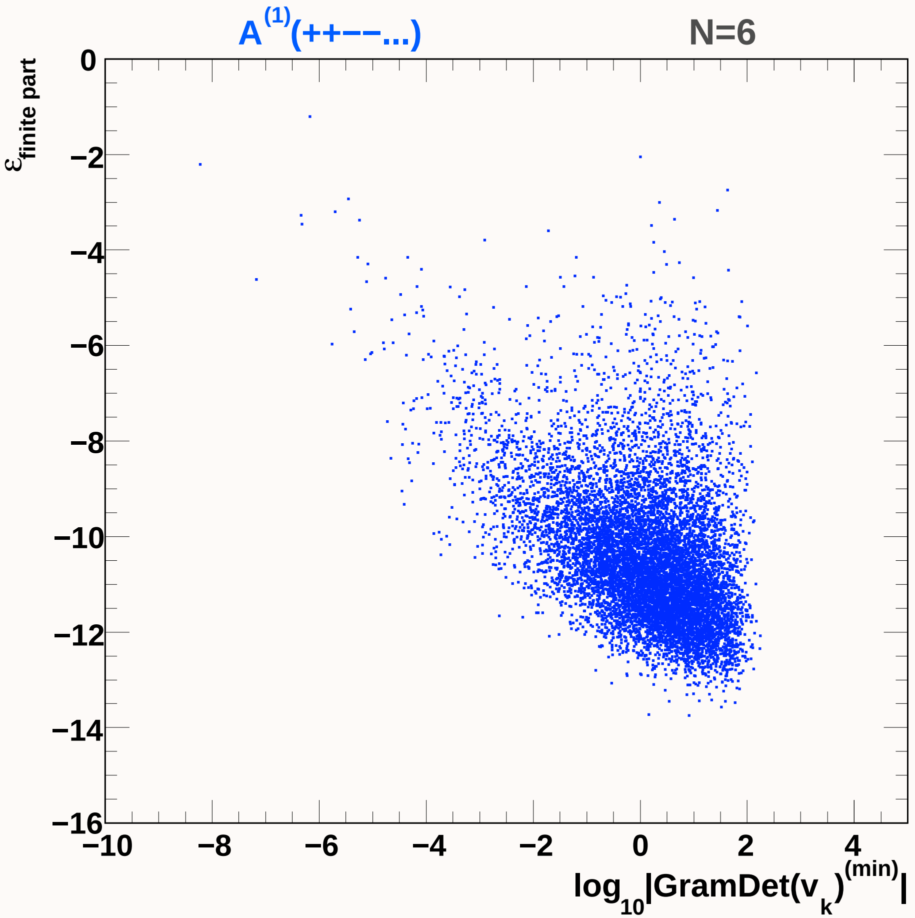


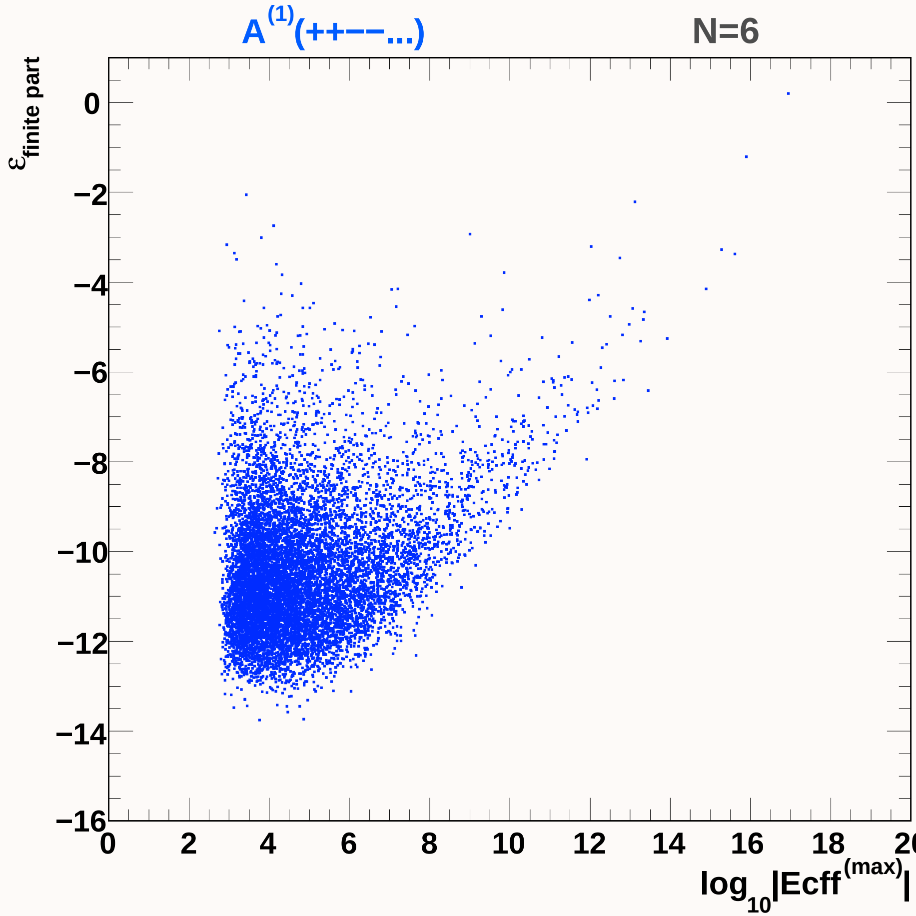


Correlations

(preliminary) (all calculations in double precision only) [Giele, Winter, arXiv:0902.0094]

- precision of finite term partly correlated with smallness/largeness of Gram dets/coefficients
- still other denominators that can become small
- e.g. the leftover d_j in the subtraction terms (even when coefficients are not large)





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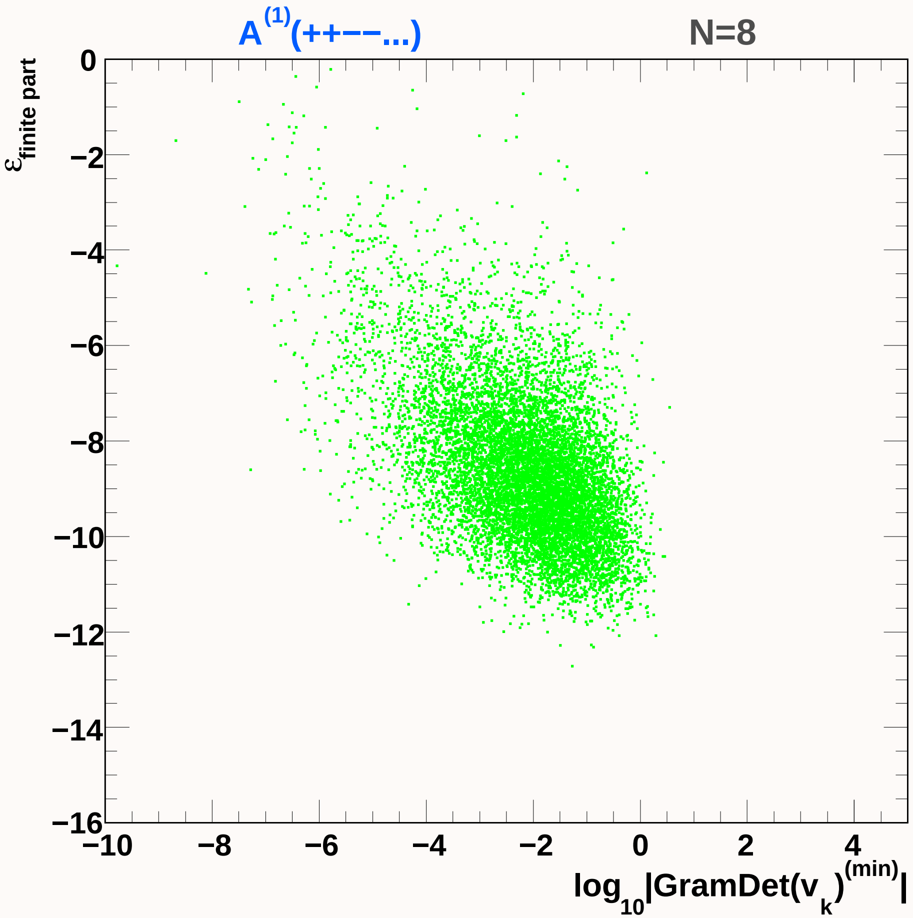

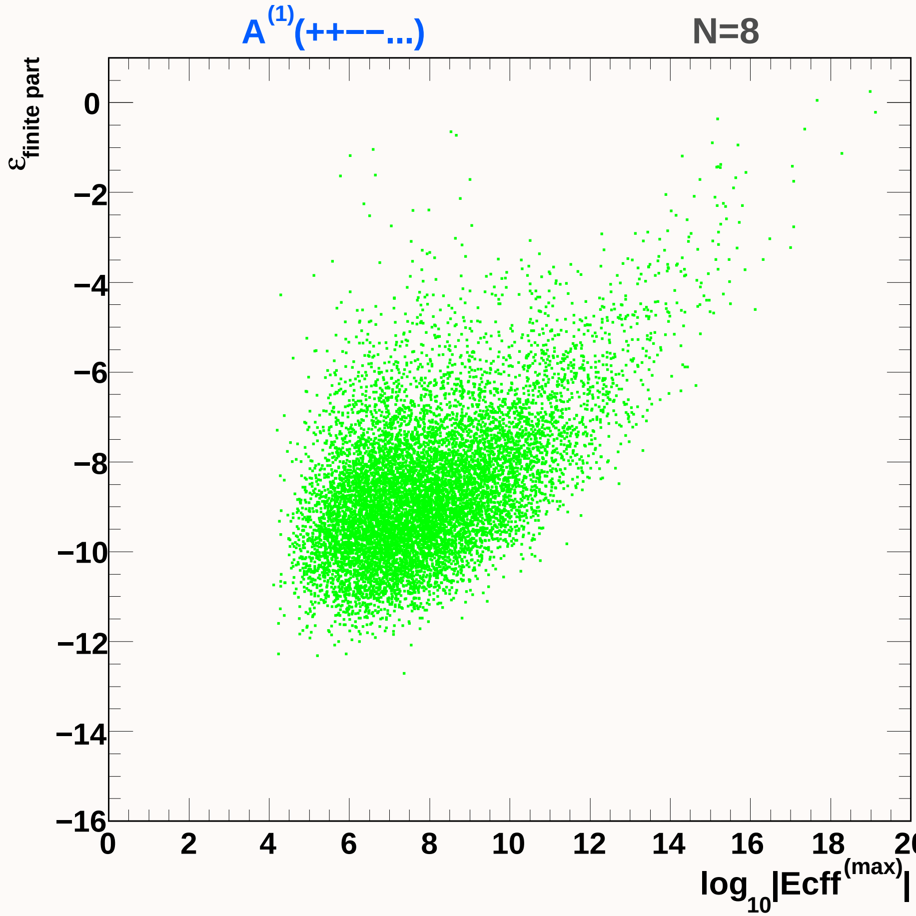

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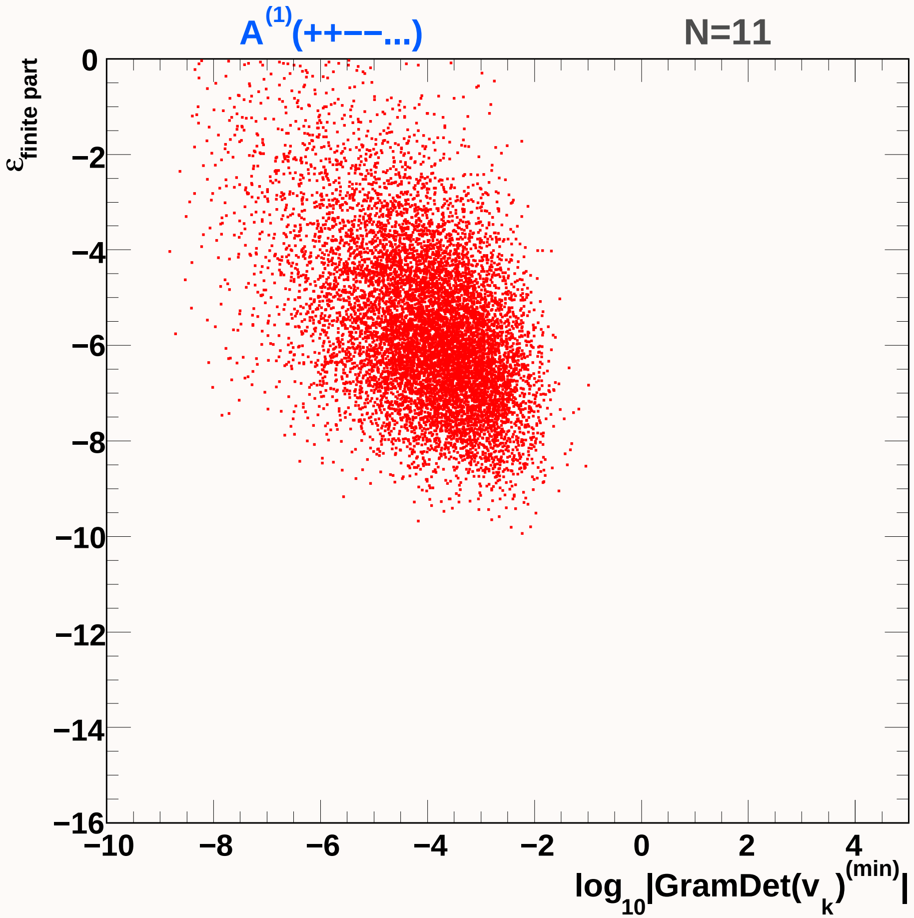
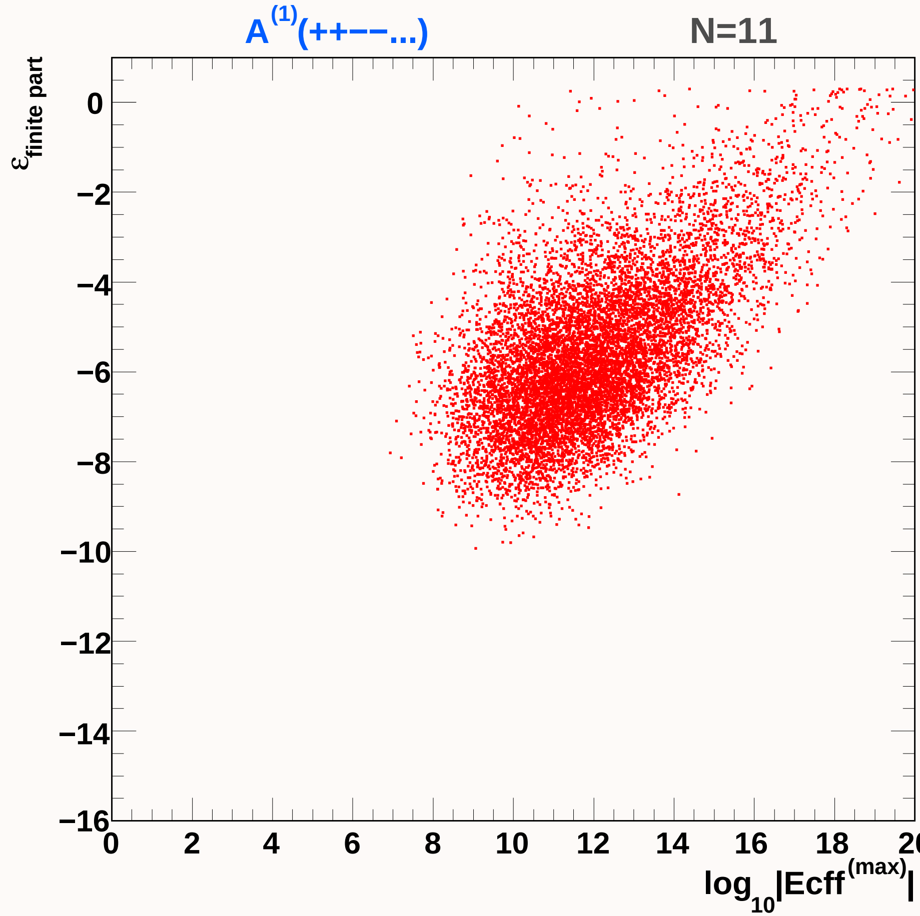
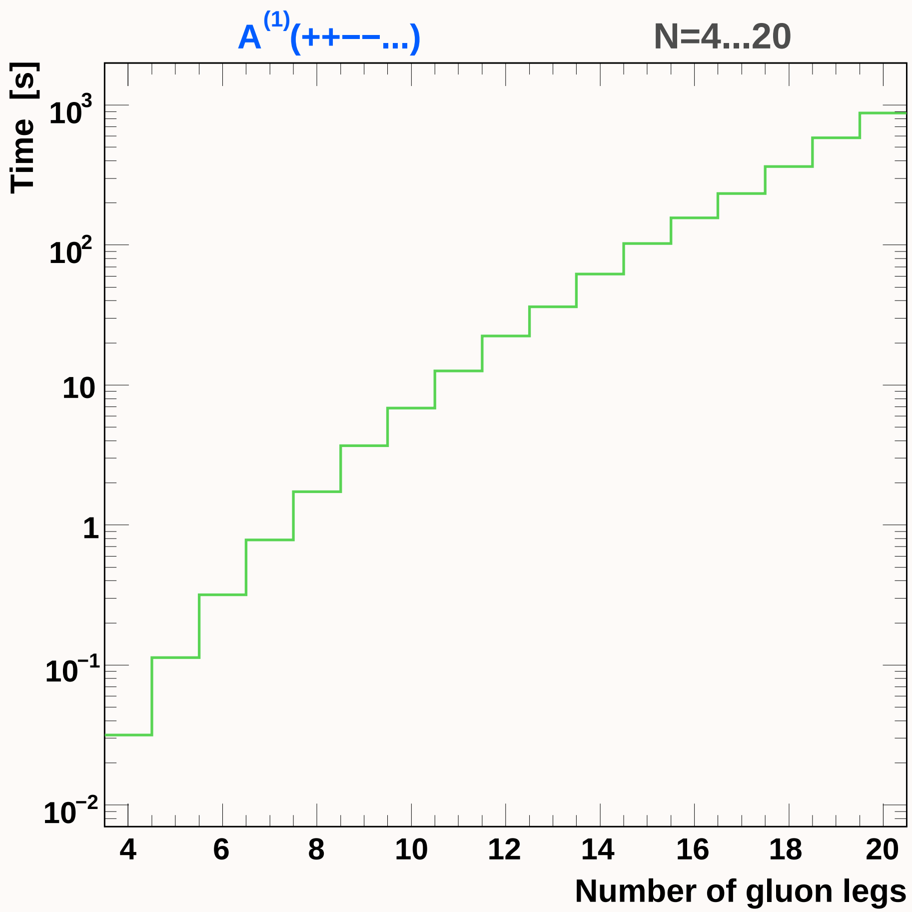


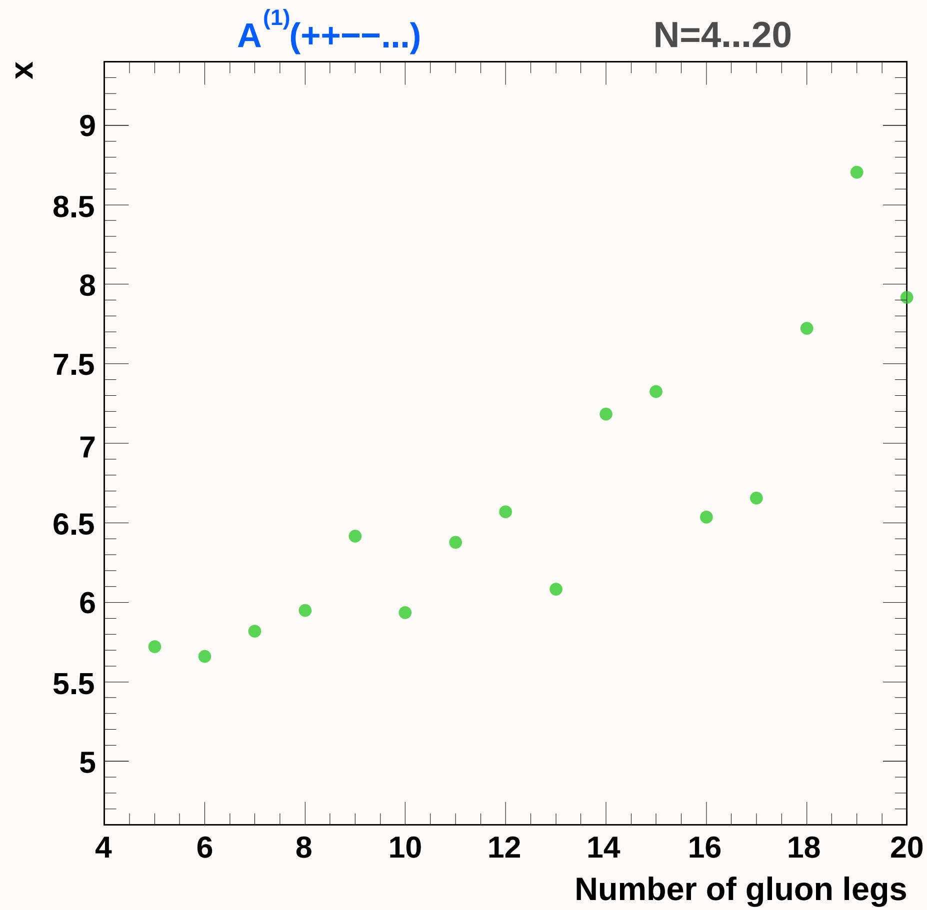


Speed of the calculation

(preliminary) (all calculations in double precision only) [Giele, Winter, arXiv:0902.0094]

- check for algorithm of polynomial complexity ($\tau \sim N^x$)
- check fractions: $x = \ln \frac{\tau_{N+1}}{\tau_N} / \ln \frac{N+1}{N}$





Colour decomposition vs. full amplitude

➡ ***factorization of one-loop amplitude in colour factors and primitive amplitudes is systematic***

- n gluons, leading-colour contributions ... making use of the symmetry of phase space

$$\int d\,PS\,|\mathcal{M}|^2 \;=\; \sum_{\text{Perm}} \int d\,PS\,|A|^2 \;\approx\; (n-1)! \int d\,PS\,|A|^2$$

- one-loop n-gluon amplitudes in $SU(N_c)$ gauge theory

$$A \;=\; g^n\, N_c \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(\lambda^{a_{\sigma_1}}\cdots\lambda^{a_{\sigma_n}})A^{[1]}_{n;1}(\sigma_1,\ldots,\sigma_n) \;+$$

$$g^n \sum_{c=2}^{\mathrm{int}(n/2)+1} \sum_{\sigma\in S_n/S_{n;c}} \mathrm{Tr}(\lambda^{a_{\sigma_1}}\cdots\lambda^{a_{\sigma_{c-1}}})\mathrm{Tr}(\lambda^{a_{\sigma_c}}\cdots\lambda^{a_{\sigma_n}})A_{n;c}(\sigma_1,\ldots,\sigma_n) \;+$$

$$g^n\, n_f \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(\lambda^{a_{\sigma_1}}\cdots\lambda^{a_{\sigma_n}})A^{[1/2]}_{n;1}(\sigma_1,\ldots,\sigma_n)$$

- obtaining ***full*** $|\mathcal{M}|^2$ ***... complicated***

  necessary to know all orderings at the same time

- (N-1)! growth in complexity of calculation ?

Construction of an NLO generator

- ***COMIX ... SM tree-level ME generator based on colour-dressed generalized BG recursions*** [GLEISBERG, HÖCHE]
  - calculates colour-dressed tree-level amplitudes
  - exponential growth

- ***Use as LO generator in the numerical generalized-unitarity method***
  - gives colour-dressed one-loop amplitudes for gluons, quarks, ..., basically the whole SM
  - truly "automated" generation of virtual corrections seems feasible
  - ⇒ full NLO MC for arbitrary SM processes when augmented by a phase-space and bremsstrahlung generator
  - e.g. combining it with automated Catani–Seymour subtraction by Gleisberg/Krauss ...

Colour-dressed recursion relations

- Colour-flow decomposition for gluon currents

$$J_\mu^{IJ}(1,2,..,n) = \sum_{\sigma\in S_n} \delta^I_{j_{\sigma_1}} \delta^{i_{\sigma_1}}_{j_{\sigma_2}} \cdots \delta^{i_{\sigma_{n-1}}}_{j_{\sigma_n}} \delta^{i_{\sigma_n}}_J \; J_\mu(\sigma_1,\sigma_2,..,\sigma_n)$$

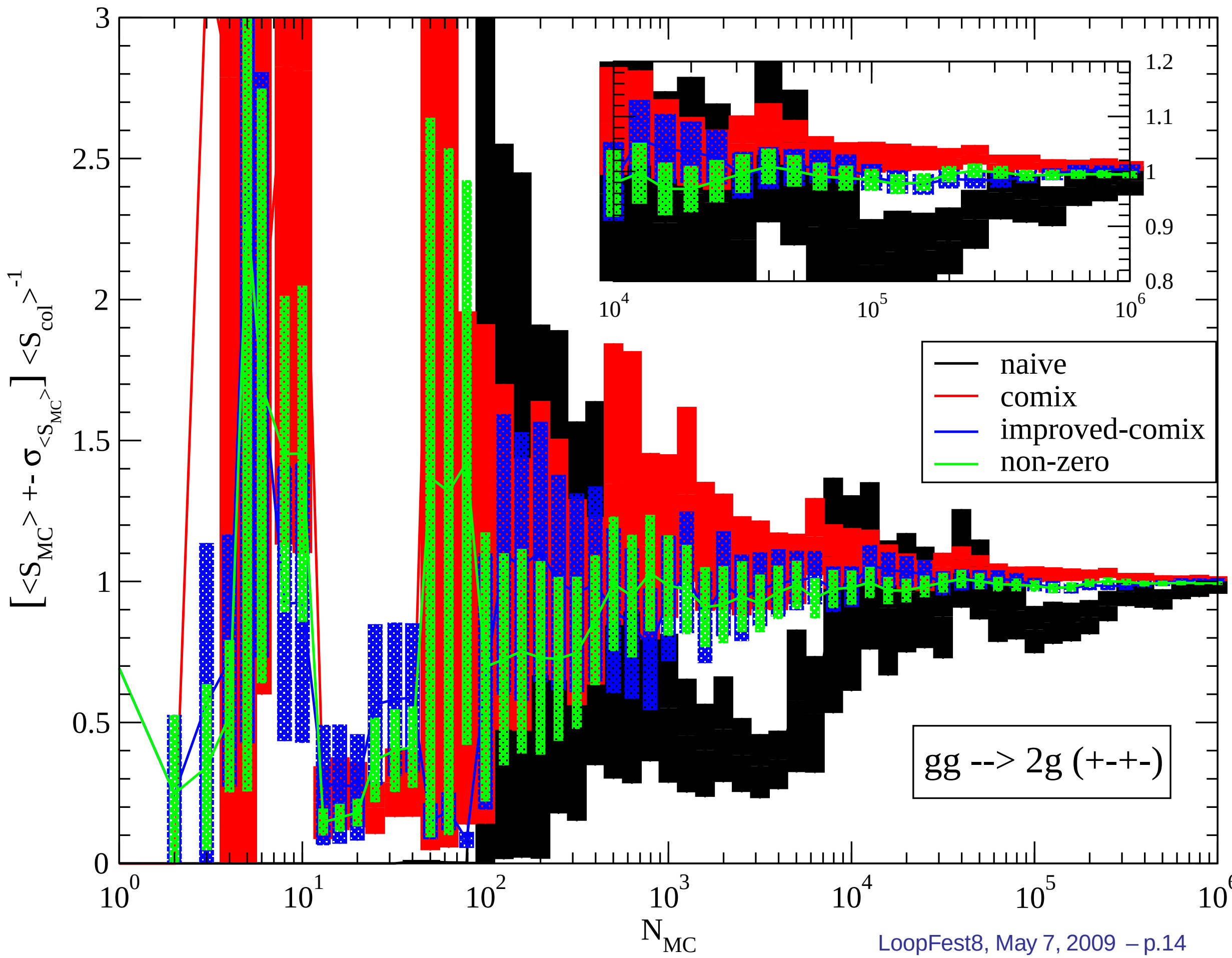
$$= \kappa^{-2}(1,2,..,n) \Big[\sum_{P_{\pi_1\pi_2}} (\delta^I_K \delta^L_M \delta^N_J - \delta^I_M \delta^N_K \delta^L_J) \, [J_\mu^{KL}(\pi_1), J_\mu^{MN}(\pi_2)] \; +$$

$$\sum_{P_{\pi_1\pi_2\pi_3}} (\delta^{ILNP}_{KMOJ} + \delta^{IPNL}_{OMKJ} - \delta^{ILPN}_{KOMJ} - \delta^{INPL}_{MOKJ}) (\{J_\mu^{KL}(\pi_1), J_\mu^{MN}(\pi_2), J_\mu^{OP}(\pi_3)\} + \pi_1 \leftrightarrow \pi_2) \Big]$$

- tree-level amplitude calculation scales as 4^N
  (replacing V_{gggg} by effective V_{ggg} or removing it gives 3^N)

- Convergence test of different colour-sampling approaches:
  check sample sum vs. exact colour sum averaged over phase space

- cf. [DUHR, HÖCHE, MALTONI]



Algorithm for full one-loop amplitudes

➡ ***algorithm used for evaluating colour-ordered one-loop amplitudes***

needs to be extended

- input: choosing external momenta, polarizations and explicit colours of external partons

  outputting: amplitude in form of complex number

- all sums over ordered cuts in the decomposition change into sums over partitions including non-cyclic, non-reflective permutations of the partition list

$$\sum_{[i_1|i_k]} \quad \rightarrow \quad \sum_{RP_{\pi_1 \cdots \pi_k}(1,2,..,n)}$$

  number of total partitions given by $\max(1, (k-1)!/2)\, \mathcal{S}_2(n,k)$

  $\Rightarrow$ considerable increase in number of terms needed to evaluate one phase-space point

- calculation of residues requires not only to sum over internal polarizations but also over internal colours

Summary

- C++ code that implements Ellis–Giele–Kunszt–Melnikov method of calculating colour-ordered one-loop amplitudes using unitarity cuts.
  - ⇒ good double-precision results for gluon case.
  - ⇒ potential improvements: fitting coefficients, higher precision.
- Outline of the algorithm for full amplitudes using colour-dressed recursion relations.
  - ⇒ algorithm is of exponential complexity.
  - ⇒ asymptotic scaling of 5^N can be expected for many legs.
  - ⇒ once proven to work, more to do: fully include quarks, squared amplitudes.
- First numerical results for colour-dressed one-loop amplitudes should be soon available allowing for a genuine assessment of the approach.