Factorization Constraints on the Structure of IR Singularities

Matthias Neubert
Johannes Gutenberg University Mainz

LoopFest VIII, Madison, 8 May 2009

Thomas Becher & MN: arXiv:0901.0722 (PRL), 0903.1126, 0904.1021 (PRD)
& work in progress with Andrea Ferroglia, Ben Pecjak, Li-Lin Yang

Outline

- Conjecture for all-order form of IR singularities in massless, non-abelian gauge theory amplitudes:
  see talk by T. Becher
  - can be absorbed into multiplicative **Z** factor, governed by anomalous dimension **Γ**
  - **Γ** involves only two-parton correlations
- Will discuss constraints on **Γ** from non-abelian exponentiation, soft-collinear factorization, and collinear limits
- Diagrammatic analysis to 3 loops, and exclusion of higher Casimir invariants at 4 loops
- Extension to massive partons (at 2 loops)

Reminder: Conjecture for $\mathbf{\Gamma}$

- All-order form:

color charges

anom. dimensions, known to three-loop order

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \, \gamma_{\rm cusp}(\alpha_s) \, \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

sum over pairs i≠j of partons

$(p_i + p_j)^2$

- SCET decoupling transformation implies:

trivial color structure

$$\mathbf{\Gamma}(s_{ij}) = \mathbf{\Gamma}_s(\Lambda_{ij}^2) + \sum_i \Gamma_c^i(M_i^2)\,\mathbf{1}$$

M_i dependence must cancel!

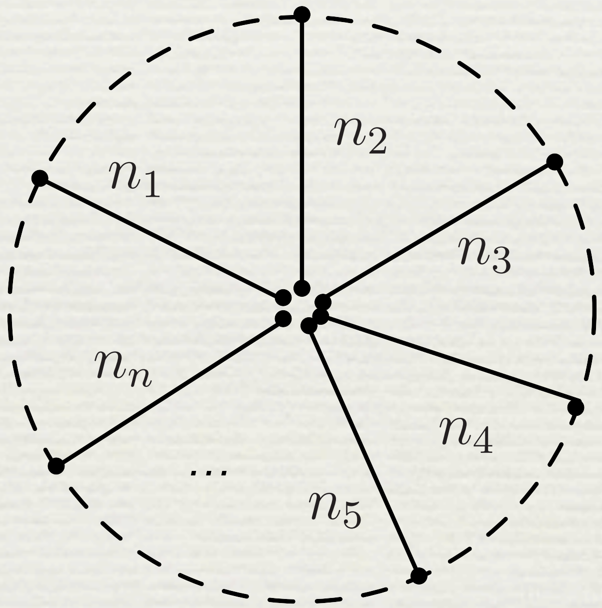
  - follows that $\mathbf{\Gamma}$ and $\mathbf{\Gamma}_s$ have same color structure

Reminder: Conjecture for $\boldsymbol{\Gamma}$

- SCET decoupling transformation removes soft interactions among collinear fields and absorbs them into soft Wilson lines

$n_i \sim p_i$ light-like reference vector

$$\boldsymbol{S}_i = \mathbf{P} \exp \left[ig \int_{-\infty}^{0} dt \, n_i \cdot A_a(tn_i) \, T_i^a \right]$$



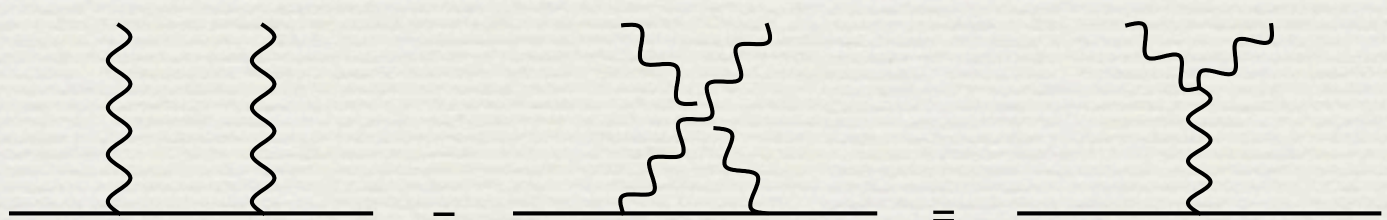
- Γ_s is anomalous dimension of n-jet Wilson-line operator:

$$\mathcal{S}(\{\underline{n}\}, \mu) = \langle 0 | \boldsymbol{S}_1(0) \dots \boldsymbol{S}_n(0) | 0 \rangle = \exp(\tilde{\mathcal{S}}(\{\underline{n}\}, \mu))$$

Non-abelian exponentiation

Gatheral 1983; Frenkel and Taylor 1984

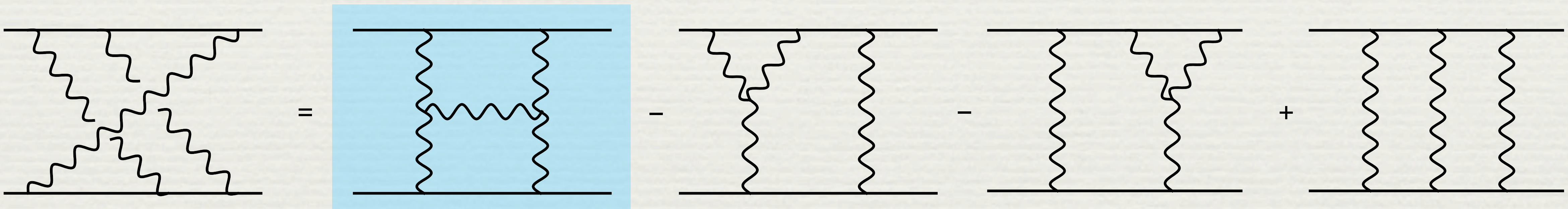
- Purely virtual amplitudes in eikonal (i.e., soft-gluon) approximation can be written as exponentials of simpler quantities, which receive contributions only from Feynman diagrams whose color weights are "color-connected" (or "maximally non-abelian")
- Color-weight graphs associated with each Feynman diagram can be simplified using the Lie commutator relation:



$$\boldsymbol{T}^a\boldsymbol{T}^b \;-\; \boldsymbol{T}^b\boldsymbol{T}^a \;=\; if^{abc}\boldsymbol{T}^c$$

Non-abelian exponentiation

- Any color-weight graph can be decomposed into a sum over products of connected webs, defined as a connected set of gluon lines (not counting crossed lines as being connected)



single connected web
"maximally nonabelian"

- Only color structures consisting of a single connected web contribute to the exponent $\tilde{S}$

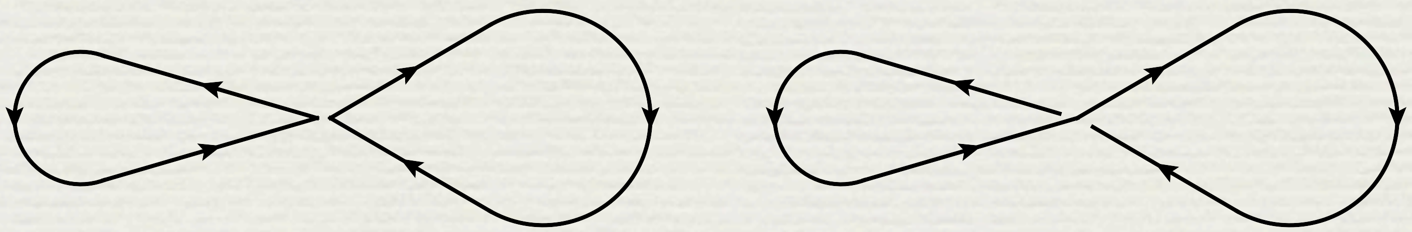
Renormalization of Wilson loops

- Wilson loops containing singular points (cusps or cross points) require UV subtractions

Polyakov 1980; Brandt, Neri, Sato 1981

- For single cusp formed by tangent vectors n_1 and n_2, renormalization factor depends on cusp angle β_{12} defined as $\cosh\beta_{12} = \dfrac{n_1 \cdot n_2}{\sqrt{n_1^2\, n_2^2}}$

- More generally, sets of related Wilson loops mix under renormalization, with $\mathbf{Z}_s$ matrix depending on all relevant cusp angles



Light-like Wilson lines

- For large values of cusp angle β_{12}, anomalous dimension associated with a cusp or cross point grows linearly with β_{12}, which is then approximately equal to $\ln(2n_1 \cdot n_2/\sqrt{n_1^2\, n_2^2})$

Korchemsky, Radyushkin 1987

- Cusp angle diverges when one or both segments approach the light-cone:

$$\Gamma(\beta_{12}) \stackrel{n_{1,2}^2 \to 0}{\longrightarrow} \Gamma_{\text{cusp}}^i(\alpha_s)\, \ln \frac{\mu^2}{\Lambda_s^2} + \ldots$$

Korchemskaya, Korchemsky 1992

- Presence of single logarithm characteristic for Sudakov problems (double logs)

Light-like Wilson lines

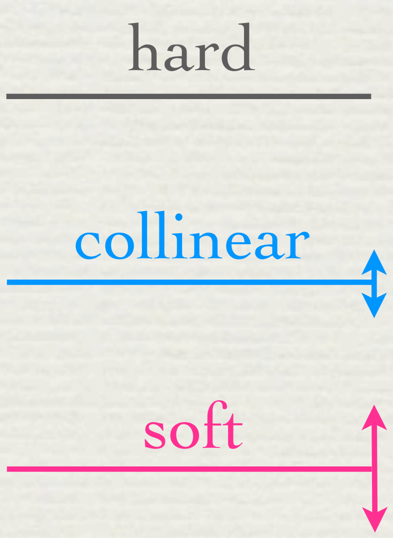
- In SCET, this feature has been found for 2-jet operators of quarks and gluons:

Manohar 2003
Becher, MN 2006
Ahrens, Becher, MN, Yang 2008

$$\Gamma_{2-\text{jet}} = -\Gamma^i_{\text{cusp}}(\alpha_s)\,\ln\frac{\mu^2}{-s} + 2\gamma^i(\alpha_s)$$

- Appearance of logarithms of hard scale is perplexing, but can be understood based on scale correlation $\mu_c^2 \sim \mu_h\,\mu_s$, which implies:

$$\ln\frac{\mu^2}{\mu_h^2} = 2\ln\frac{\mu^2}{\mu_c^2} - \ln\frac{\mu^2}{\mu_s^2}$$



- For such a rewriting to be possible, the anomalous dimension must depend single-logarithmically on momenta

Light-like Wilson lines

- Introducing IR regulators $p_i^2 \neq 0$ to define the soft and collinear scales, we obtain:

$$\beta_{ij} = L_i + L_j - \ln \frac{\mu^2}{-s_{ij}}$$

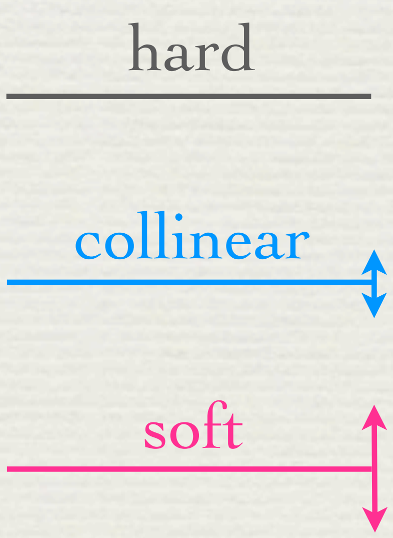
hard log

$$\beta_{ij} = \ln \frac{-s_{ij}\,\mu^2}{(-p_i^2)(-p_j^2)}$$

soft log

$$L_i = \ln \frac{\mu^2}{-p_i^2}$$

collinear log



Soft anomalous-dimension matrix

- Decompositions:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \mathbf{\Gamma}_s(\{\underline{\beta}\},\mu) + \sum_i \Gamma_c^i(L_i,\mu)$$

$$\Gamma_c^i(L_i) = -\Gamma_{\text{cusp}}^i(\alpha_s)\, L_i + \gamma_c^i(\alpha_s)$$

- Key equation: see also: Gardi, Magnea, arXiv:0901.1091

$$\frac{\partial \mathbf{\Gamma}_s(\{\underline{s}\},\{\underline{L}\},\mu)}{\partial L_i} = \Gamma_{\text{cusp}}^i(\alpha_s)$$

- Enforces linearity in cusp angles β_{ij} (with one exception, see below) and significantly restricts color structures

Soft anomalous-dimension matrix

- Only exception would be a more complicated dependence on conformal cross ratios, which are independent of collinear scales:

$$\beta_{ijkl} = \beta_{ij} + \beta_{kl} - \beta_{ik} - \beta_{jl} = \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

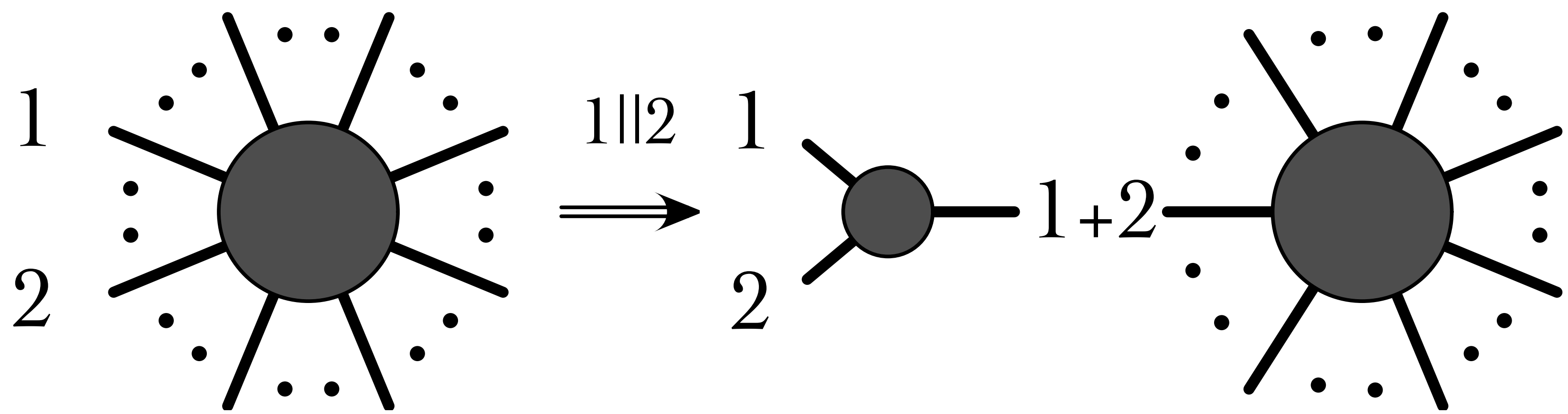
Gardi, Magnea 2009

- Any polynomial dependence on such ratios can be excluded using arguments based on consistency with collinear limits

Consistency with collinear limits

- When two partons become collinear, an n-point amplitude M_n reduces to an (n-1)-parton amplitude times a splitting function: Berends, Giele 1989; Mangano, Parke 1991 Kosower 1999; Catani, de Florian, Rodrigo 2003

$$|\mathcal{M}_n(\{p_1, p_2, p_3, \dots, p_n\})\rangle = \mathbf{Sp}(\{p_1, p_2\})\, |\mathcal{M}_{n-1}(\{P, p_3, \dots, p_n\})\rangle + \dots$$



$$\mathbf{\Gamma}_{\mathrm{Sp}}(\{p_1, p_2\}, \mu) = \mathbf{\Gamma}(\{p_1, \dots, p_n\}, \mu) - \mathbf{\Gamma}(\{P, p_3 \dots, p_n\}, \mu)\big|_{\boldsymbol{T}_P \to \boldsymbol{T}_1 + \boldsymbol{T}_2}$$

- $\mathbf{\Gamma}_{\mathrm{Sp}}$ must be independent of momenta and colors of partons 3, ..., n

Becher, MN 2009

Consistency with collinear limits

- The form we propose is consistent with factorization in the collinear limit:

$$\mathbf{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) = \mathbf{\Gamma}(\{p_1,\dots,p_n\},\mu) - \mathbf{\Gamma}(\{P,p_3\dots,p_n\},\mu)\big|_{\boldsymbol{T}_P\to\boldsymbol{T}_1+\boldsymbol{T}_2}$$

$$\mathbf{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) = \gamma_{\rm cusp}\left[\boldsymbol{T}_1\cdot\boldsymbol{T}_2\,\ln\frac{\mu^2}{-s_{12}} + \boldsymbol{T}_1\cdot(\boldsymbol{T}_1+\boldsymbol{T}_2)\,\ln z + \boldsymbol{T}_2\cdot(\boldsymbol{T}_1+\boldsymbol{T}_2)\,\ln(1-z)\right] + \gamma^1 + \gamma^2 - \gamma^P\,,$$

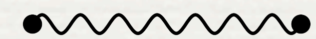
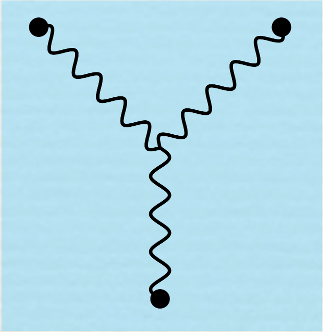
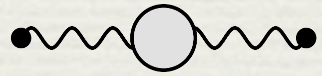
momentum fractions of partons 1, 2

- But this would not work if $\mathbf{\Gamma}$ would involve terms of higher powers in color generators $\boldsymbol{T}_i$ or momentum variables
- A strong, new constraint!

$$\mathbf{\Gamma}_s(\{\underline{\beta}\},\mu) \stackrel{?}{=} -\sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\text{cusp}}(\alpha_s)\,\beta_{ij} + \sum_i \gamma_s^i(\alpha_s)$$

Diagrammatic analysis of the soft anomalous-dimension matrix

Order-by-order analysis

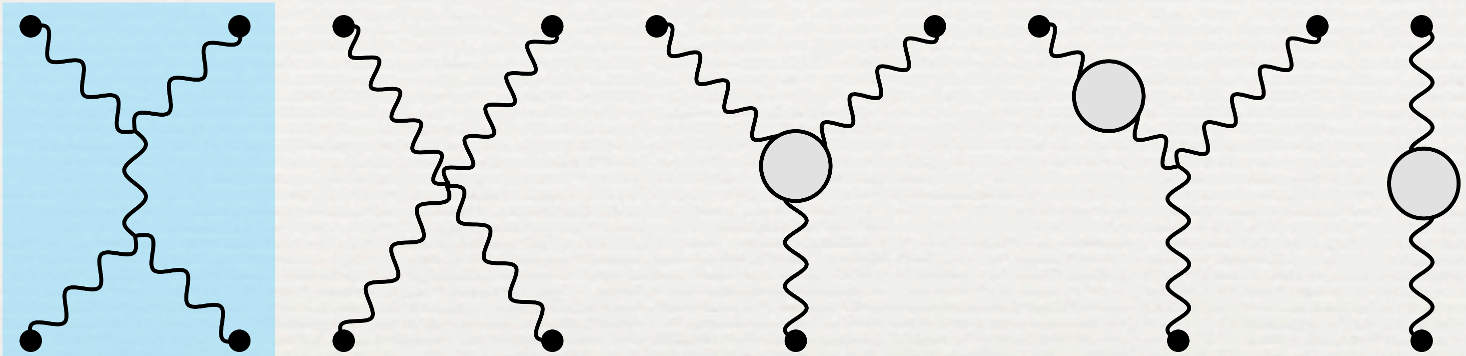
- One loop (recall $\sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j = -\sum_i \boldsymbol{T}_i^2 = -\sum_i C_i$)
  - one leg: $\boldsymbol{T}_i^2 = C_i$ 
  - two legs: $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$
- Two loops 
  - one leg: $-if^{abc}\,\boldsymbol{T}_i^a\,\boldsymbol{T}_i^b\,\boldsymbol{T}_i^c = \frac{C_A C_i}{2}$
  - two legs: $-if^{abc}\,\boldsymbol{T}_i^a\,\boldsymbol{T}_i^b\,\boldsymbol{T}_j^c = \frac{C_A}{2}\boldsymbol{T}_i \cdot \boldsymbol{T}_j$ (only new structure) 
  - three legs: $-if^{abc}\,\boldsymbol{T}_i^a\,\boldsymbol{T}_j^b\,\boldsymbol{T}_k^c$

$\Rightarrow$ vanishes, since no antisymmetric momentum structure in i,j,k consistent with soft-collinear factorization exists!

explains cancellations observed in:
Mert Aybat, Dixon, Sterman 2006; Dixon 2009

Three-loop order

- Single webs:



(only new structure)

- Six new structures consistent with non-abelian exponentiation exist, two of which are compatible with soft-collinear factorization:

$$\begin{aligned}\boldsymbol{\Delta\Gamma}_3(\{\underline{p}\},\mu) &= -\frac{\bar{f}_1(\alpha_s)}{4}\sum_{(i,j,k,l)} f^{ade} f^{bce}\,\boldsymbol{T}_i^a\,\boldsymbol{T}_j^b\,\boldsymbol{T}_k^c\,\boldsymbol{T}_l^d\,\ln\frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}\\ &\quad -\bar{f}_2(\alpha_s)\sum_{(i,j,k)} f^{ade} f^{bce}\,\big(\boldsymbol{T}_i^a\,\boldsymbol{T}_i^b\big)_+\,\boldsymbol{T}_j^c\,\boldsymbol{T}_k^d\,,\end{aligned}$$

more generally, arbitrary odd function of conformal cross ratio

Three-loop order

- Neither of these is compatible with collinear limits: the splitting function would depend on colors and momenta of the additional partons
- Consider, e.g., the second term:

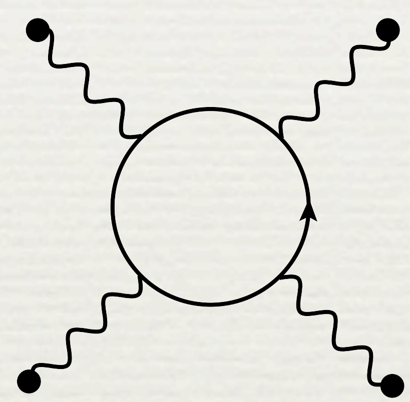
$$\mathbf{\Delta\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu)\big|_{\bar{f}_2(\alpha_s)} = 2 f^{ade} f^{bce} \left[\left(\boldsymbol{T}_1^a \boldsymbol{T}_1^b\right)_+ \left(\boldsymbol{T}_2^c \boldsymbol{T}_2^d\right)_+ - \sum_{i\neq 1,2} \left(\boldsymbol{T}_1^a \boldsymbol{T}_2^b + \boldsymbol{T}_2^a \boldsymbol{T}_1^b\right) \left(\boldsymbol{T}_i^c \boldsymbol{T}_i^d\right)_+ \right]$$

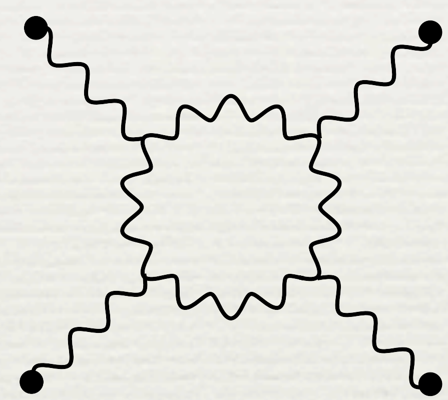
$$\mathbf{\Delta\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu)\big|_{\bar{f}_1(\alpha_s)} = f^{ade} f^{bce} \sum_{(i,j)\neq 1,2} \left(\boldsymbol{T}_1^a \boldsymbol{T}_2^b + \boldsymbol{T}_2^a \boldsymbol{T}_1^b\right) \boldsymbol{T}_i^c \boldsymbol{T}_j^d \ln \frac{\mu^2}{-s_{ij}} + \ \dots$$

dependence on color invariants and momenta of additional partons (i≠1,2)

Four-loops and beyond

- Interesting new webs involving higher Casimir invariants first arise at four loops





$$d_F^{abcd}\, \boldsymbol{T}_i^a\, \boldsymbol{T}_j^b\, \boldsymbol{T}_k^c\, \boldsymbol{T}_l^d = d_F^{abcd}\left(\boldsymbol{T}_i^a\, \boldsymbol{T}_j^b\, \boldsymbol{T}_k^c\, \boldsymbol{T}_l^d\right)_+$$

$$d_R^{a_1 a_2 \ldots a_n} = \mathrm{tr}\left[\,(\boldsymbol{T}_R^{a_1}\, \boldsymbol{T}_R^{a_2} \ldots \boldsymbol{T}_R^{a_n})_+\right]$$

- One linear combination of such terms would be compatible with soft-collinear factorization, but does not have the correct collinear limit

Casimir scaling

- Applied to the two-jet case (form factors), our formula thus implies Casimir scaling of the cusp anomalous dimension:

$$\frac{\Gamma^q_{\text{cusp}}(\alpha_s)}{C_F} = \frac{\Gamma^g_{\text{cusp}}(\alpha_s)}{C_A} = \gamma_{\text{cusp}}(\alpha_s)$$

- Checked explicitly at three loops Moch, Vermaseren, Vogt 2004
- But contradicts expectations from AdS/CFT correspondence (high-spin operators in strong-coupling limit)

Armoni 2006
Alday, Maldacena 2007

- Presumably not a real conflict ...

Wanted: 3- and 4-loop checks

- Full three-loop 4-jet amplitudes in N=4 super Yang-Mills theory were expressed in terms of small number of scalar integrals Bern et al. 2008
- Once these can be calculated, this will provide stringent test of our arguments (note recent calculation of three-loop form-factor integrals)
  Baikov et al. 2009;
  Heinrich, Huber, Kosower, Smirnov 2009
- Calculation of four-loop cusp anomalous dimension would provide non-trivial test of Casimir scaling, which is then no longer guaranteed by non-abelian exponentiation

Extension to massive partons

Processes with heavy particles

- Have extended our analysis to amplitudes which include massive partons Becher, MN, arXiv:0904.1021
- Effective theory is combination of HQET (for heavy partons) and SCET (massless partons)
- Soft function contains both massless and timelike Wilson lines:

$$\mathcal{S}(\{\underline{n}\},\{\underline{v}\},\mu) = \langle 0|\boldsymbol{S}_{n_1}\dots\boldsymbol{S}_{n_k}\,\boldsymbol{S}_{v_{k+1}}\dots\boldsymbol{S}_{v_n}|0\rangle$$

  - v_i are 4-velocities of the massive partons
  - n_i are light-light reference vectors

Processes with heavy particles

- Both the full and the effective theory know about the 4-velocities of the massive partons
- Therefore much weaker constraints hold for the massive case:
  - no soft-collinear factorization
  - no constraint from (quasi-)collinear limits
- For the purely massive case, all structures allowed by non-abelian exponentiation at a given order will be present!

Anomalous dimension to two loops

- One- and two-parton terms:

$$
\begin{aligned}
\boldsymbol{\Gamma}(\{\underline{p}\},\{\underline{m}\},\mu)\big|_{2-\text{parton}}
&= \sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\text{cusp}}(\alpha_s)\,\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)\\
&\quad - \sum_{(I,J)} \frac{\boldsymbol{T}_I\cdot\boldsymbol{T}_J}{2}\,\gamma_{\text{cusp}}(\beta_{IJ},\alpha_s) + \sum_I \gamma^I(\alpha_s)\\
&\quad + \sum_{I,j} \boldsymbol{T}_I\cdot\boldsymbol{T}_j\,\gamma_{\text{cusp}}(\alpha_s)\,\ln\frac{m_I\mu}{-s_{Ij}}\,,
\end{aligned}
$$

massless partons

massive partons

new!

- Generalizes structure found for massless case
- Reproduces IR poles of QCD amplitudes after appropriate matching of coupling constants

Anomalous dimension to two loops

- New ingredients $\gamma_{\text{cusp}}(\beta_{IJ},\alpha_s)$ and $\gamma^I(\alpha_s)$ can be extracted from known results for heavy-heavy and heavy-light form factors
- In particular:

$$\gamma_{\text{cusp}}(\beta,\alpha_s) = \gamma_{\text{cusp}}(\alpha_s)\,\beta\coth\beta + C_A\left(\frac{\alpha_s}{\pi}\right)^2\Bigg[\frac{\zeta_3}{2}+\frac{\pi^2}{24}(\beta\coth\beta-1)-\coth\beta\int_0^\beta d\psi\,\psi\coth\psi$$
$$-\frac{\sinh 2\beta}{2}\int_0^\beta d\psi\,\frac{\psi\coth\psi-1}{\sinh^2\beta-\sinh^2\psi}\ln\frac{\sinh\beta}{\sinh\psi}+\coth^2\beta\int_0^\beta d\psi\,\psi(\beta-\psi)\coth\psi\Bigg]+\dots$$

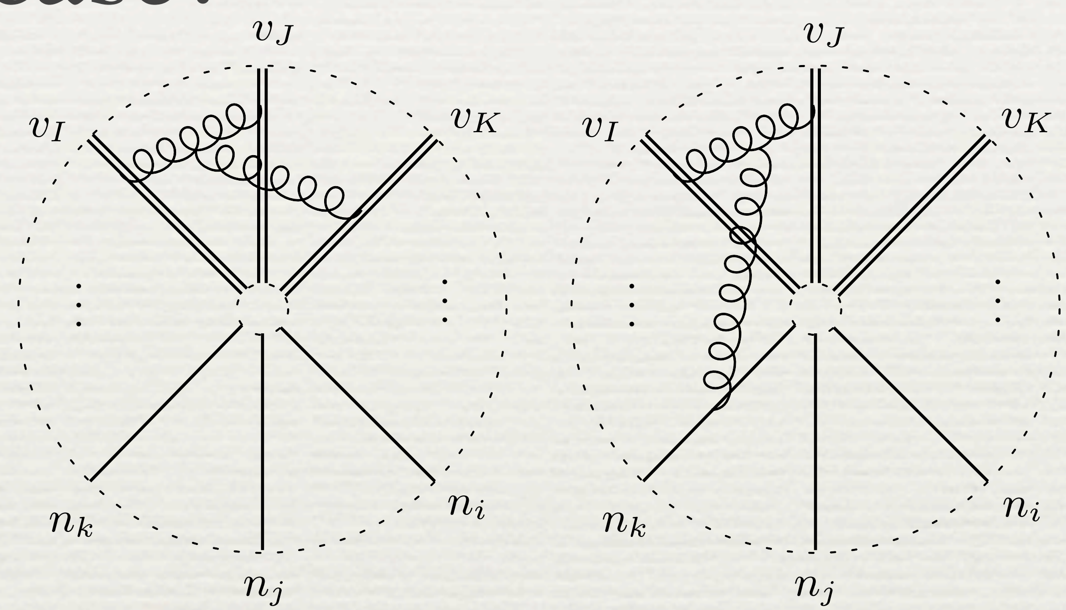
derived using:
Korchemsky, Radyushkin 1987, 1992
MN 2004
(see also: Kidonakis 2009)

- However ...

Anomalous dimension to two loops

- ... in addition also 3-parton correlations appear in massless case!

Mitov, Sterman, Sung 2009



- General structure [with $\beta_{IJ} = \text{arccosh}(v_I \cdot v_J)$]:

Becher, MN 2009

$$\mathbf{\Gamma}(\{\underline{p}\}, \{\underline{m}\}, \mu)\big|_{3-\text{parton}}$$
$$= if^{abc} \sum_{(I,J,K)} \boldsymbol{T}_I^a \boldsymbol{T}_J^b \boldsymbol{T}_K^c \, F_1(\beta_{IJ}, \beta_{JK}, \beta_{KI})$$
$$+ if^{abc} \sum_{(I,J)} \sum_k \boldsymbol{T}_I^a \boldsymbol{T}_J^b \boldsymbol{T}_k^c \, f_2\Big(\beta_{IJ}, \ln\frac{-\sigma_{Jk}\, v_J \cdot p_k}{-\sigma_{Ik}\, v_I \cdot p_k}\Big)$$

- both structures vanish when two velocities coincide
- no correlations involving a single heavy parton

Limit of small parton masses

- Particle masses then serve as IR regulators for collinear singularities
- Factorization theorems allow one to derive massive amplitudes from massless ones

Penin 2005; Moch, Mitov 2006; Becher, Melnikov 2007

- Our one- and two-parton terms are consistent with this factorization:

$$\boldsymbol{\Gamma}(\{\underline{p}\},\{\underline{m}\to 0\},\mu)\big|_{2-\mathrm{parton}} - \boldsymbol{\Gamma}(\{\underline{p}\},\{\underline{0}\},\mu)$$
$$= \sum_I \left[C_I\, \gamma_{\mathrm{cusp}}(\alpha_s)\, \ln\frac{\mu}{m_I} + \gamma^I(\alpha_s) - \gamma^i(\alpha_s) \right]$$

- But if F_1 or f_2 do not vanish in this limit, the factorization theorems require modifications

Conclusions

- IR divergences of arbitrary scattering amplitudes in gauge theories can be derived from SCET anomalous-dimension matrix $\mathbf{\Gamma}$
- Stringent constraints on $\mathbf{\Gamma}$ arise from non-abelian exponentiation (general case), and soft-collinear factorization & collinear limits (massless case only)
- Conjectured form of pure color-dipole correlations demonstrated to hold at 3- and (partial) 4-loop order, assuming polynomial dependence on β_{ijkl}
- In massive case, previously observed properties of 2-loop three-parton correlations understood from symmetry properties in effective theory