IR singularities of gauge theory amplitudes and resummation

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IR singularities

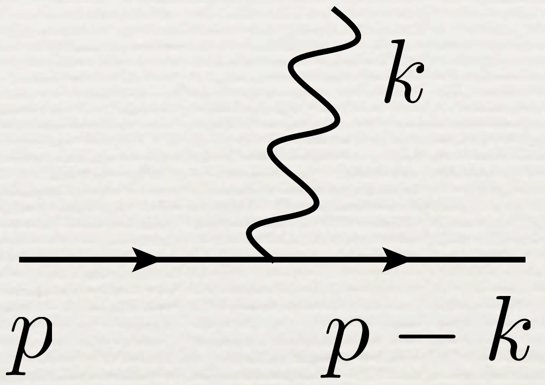
- On-shell parton scattering amplitudes in gauge theories contain IR divergences from soft and collinear loop momenta
- IR singularities cancel between real and virtual contributions

Bloch, Nordsieck 1937
Kinoshita 1962; Lee, Nauenberg 1964

- Nevertheless interesting:
  - resummation of large Sudakov logarithms remaining after cancellation of divergences (relevant for LHC physics!)
  - check on multi-loop calculations

IR singularities in QED

- Singularities arise from soft photon emission (for $m_e \neq 0$); eikonal approximation:



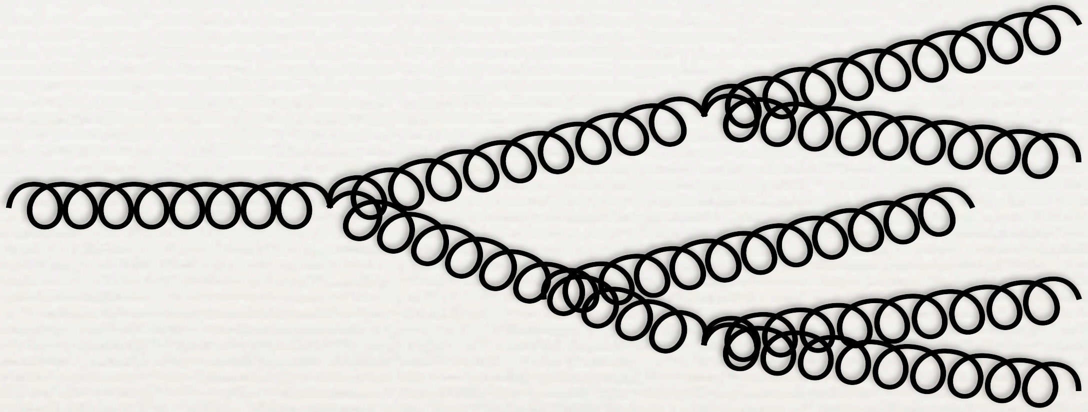
$$\cdots \frac{\not p - \not k + m}{(p-k)^2 - m^2}\,\gamma_\mu\, u(p)$$

$$\approx \ \ldots\ u(p)\,\frac{p_\mu}{p \cdot k}$$

- IR divergent part is a multiplicative factor
- Higher-order terms obtained by exponentiating leading-order soft contribution.

Yennie, Frautschi, Suura 1961
Weinberg 1965

IR singularities in QCD



> “In [Yang-Mills theory] a soft photon emitted from an external line can itself emit a pair of soft charged massless particles, which themselves emit soft photons, and so on, building up a cascade of soft massless particles each of which contributes an infra-red divergence. The elimination of such complicated interlocking infra-red divergences would certainly be a Herculean task, and might not even be possible. ”

Weinberg, Phys. Rev. 140B (1965)

IR singularities in QCD

- Much more complicated
  - soft and collinear singularities
  - gluons carry color charge, hence soft emissions do not simply exponentiate
  - but only a restricted set of higher-order contributions can appear (non-abelian exponentiation theorem) Gatheral 1983; Frenkel, Taylor 1984
- For long time, explicit form of IR poles was only understood at two-loop order Catani 1998

Overview of the talk

- IR singularities of gauge theory on-shell amplitudes
  - can be absorbed into multiplicative Z-factor, governed by an anomalous dimension $\mathbf{\Gamma}$
  - conjecture: for massless theories $\mathbf{\Gamma}$ involves only two-parton color-correlations
- Effective theory analysis
  - on-shell amplitudes as Wilson coefficients in Soft-Collinear Effective Theory
  - constraints on $\mathbf{\Gamma}$ from soft-collinear factorization
- Phenomenological application: higher-log resummation for n-jet processes.

Matthias Neubert's talk

- Constraints on $\mathbf{\Gamma}$
  - non-abelian exponentiation
  - soft-collinear factorization
  - collinear limits
- Order-by-order analysis to three loops
- Higher-Casimir terms at four loops
- Amplitudes involving massive partons

Color-space formalism

- Represent amplitudes as vectors in color space:

$$\left|c_1, c_2, \ldots, c_n\right\rangle$$

Catani, Seymour 1996

↑ color index of first parton

- Color generator for i^th parton $\boldsymbol{T}_i^a \left|c_1, c_2, \ldots, c_n\right\rangle$ acts like a matrix:
  - t^a matrix for quarks, f^{abc} for gluons
  - product $\boldsymbol{T}_i \cdot \boldsymbol{T}_j = \sum_a \boldsymbol{T}_i^a \boldsymbol{T}_j^a$ (commutative)
  - charge conservation $\sum_i \boldsymbol{T}_i^a = 0$ implies:

$$\sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j = -\sum_i \boldsymbol{T}_i^2 = -\sum_i C_i$$

i≠j

C_F or C_A

Catani's two-loop formula (1998)

("... beautiful, yet mysterious ...")

- Specifies IR singularities of dimensionally regularized n-parton amplitudes at two loops:

$$\left[1-\frac{\alpha_s}{2\pi}\boldsymbol{I}^{(1)}(\epsilon)-\left(\frac{\alpha_s}{2\pi}\right)^2\boldsymbol{I}^{(2)}(\epsilon)+\dots\right]|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle=\text{finite}$$

amplitude is vector in color space

with

$$\boldsymbol{I}^{(1)}(\epsilon)=\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\sum_i\left(\frac{1}{\epsilon^2}+\frac{g_i}{\boldsymbol{T}_i^2}\frac{1}{\epsilon}\right)\sum_{j\neq i}\frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\left(\frac{\mu^2}{-s_{ij}}\right)^\epsilon$$

$$(p_i+p_j)^2$$

$$\boldsymbol{I}^{(2)}(\epsilon)=\frac{e^{-\epsilon\gamma_E}\,\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(K+\frac{\beta_0}{2\epsilon}\right)\boldsymbol{I}^{(1)}(2\epsilon)$$

$$-\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon)\left(\boldsymbol{I}^{(1)}(\epsilon)+\frac{\beta_0}{\epsilon}\right)+\boldsymbol{H}^{(2)}_{\text{R.S.}}(\epsilon)$$

unspecified

- Later derivation using factorization properties and IR evolution equation for form factor

Sterman, Tejeda-Yeomans 2003

All-order generalization

- IR divergences in d=4-2ε can be absorbed into a multiplicative factor **Z** (a matrix in color space), which derives from an anomalous-dimension matrix:

TB, Neubert 2009

$$\overset{\text{finite}}{|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle} = \lim_{\epsilon\to 0} \boldsymbol{Z}^{-1}(\epsilon,\{\underline{p}\},\mu)\,|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle$$

$$\boldsymbol{Z}(\epsilon,\{\underline{p}\},\mu) = \mathbf{P}\exp\left[\int_\mu^\infty \frac{d\mu'}{\mu'}\,\boldsymbol{\Gamma}(\{\underline{p}\},\mu')\right]$$

- Corresponding RG evolution equation:

$$\frac{d}{d\ln\mu}\,|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle = \boldsymbol{\Gamma}(\{\underline{p}\},\mu)\,|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle$$

⇒ can be used to resum Sudakov logarithms

All-order generalization

- Anomalous dimension is conjectured to be extremely simple:

color charges

anom. dimensions,
known to three-loop order

$$\boldsymbol{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2}\, \gamma_{\text{cusp}}(\alpha_s)\, \ln \frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)$$

sum over pairs
i≠j of partons

$(p_i + p_j)^2$

  - simple structure, reminiscent of QED
  - IR poles determined by color charges and momenta of external partons
  - color dipole correlations, like at one-loop order

Z factor to three loops

- Explicit result:

d-dimensional β-function

$$\ln \boldsymbol{Z}(\epsilon, \{\underline{p}\}, \mu) = \int\limits_0^{\alpha_s} \frac{d\alpha}{\alpha} \, \frac{1}{2\epsilon - \beta(\alpha)/\alpha} \left[\boldsymbol{\Gamma}(\{\underline{p}\}, \mu, \alpha) + \int\limits_0^{\alpha} \frac{d\alpha'}{\alpha'} \, \frac{\Gamma'(\alpha')}{2\epsilon - \beta(\alpha')/\alpha'} \right]$$

where

$$\Gamma'(\alpha_s) \equiv \frac{\partial}{\partial \ln \mu} \boldsymbol{\Gamma}(\{\underline{p}\}, \mu, \alpha_s) = -\gamma_{\text{cusp}}(\alpha_s) \sum_i C_i$$

- Perturbative expansion:

all coefficients known!

$$\begin{aligned} \ln \boldsymbol{Z} \; &= \frac{\alpha_s}{4\pi} \left(\frac{\Gamma'_0}{4\epsilon^2} + \frac{\boldsymbol{\Gamma}_0}{2\epsilon} \right) + \left(\frac{\alpha_s}{4\pi} \right)^2 \left[-\frac{3\beta_0 \Gamma'_0}{16\epsilon^3} + \frac{\Gamma'_1 - 4\beta_0 \boldsymbol{\Gamma}_0}{16\epsilon^2} + \frac{\boldsymbol{\Gamma}_1}{4\epsilon} \right] \\ &+ \left(\frac{\alpha_s}{4\pi} \right)^3 \left[\frac{11\beta_0^2 \, \Gamma'_0}{72\epsilon^4} - \frac{5\beta_0 \Gamma'_1 + 8\beta_1 \Gamma'_0 - 12\beta_0^2 \, \boldsymbol{\Gamma}_0}{72\epsilon^3} + \frac{\Gamma'_2 - 6\beta_0 \boldsymbol{\Gamma}_1 - 6\beta_1 \boldsymbol{\Gamma}_0}{36\epsilon^2} + \frac{\boldsymbol{\Gamma}_2}{6\epsilon} \right] + \dots \end{aligned}$$

⇒ exponentiation yields **Z** factor at three loops!

Checks

- Expression for IR pole terms agrees with all known perturbative results:
  - 3-loop quark and gluon form factors, which determine the functions $\gamma^{q,g}(\alpha_s)$

    Moch, Vermaseren, Vogt 2005
  - 2-loop 3-jet qqg amplitude

    Garland, Gehrmann et al. 2002
  - 2-loop 4-jet amplitudes

    Anastasiou, Glover et al. 2001
    Bern, De Freitas, Dixon 2002, 2003
  - 3-loop 4-jet amplitudes in N=4 super Yang-Mills theory in planar limit

    Bern et al. 2005, 2007

Catani's result

- Comparison with Catani's formula at two loops yields explicit expression for 1/ε pole term:

$$\boldsymbol{H}^{(2)}_{\text{R.S.}}(\epsilon) = \frac{1}{16\epsilon}\sum_i \left(\gamma_1^i - \frac{1}{4}\,\gamma_1^{\text{cusp}}\,\gamma_0^i + \frac{\pi^2}{16}\,\beta_0\,C_i\right)$$

$$+ \frac{i f^{abc}}{24\epsilon}\sum_{(i,j,k)} \boldsymbol{T}_i^a\,\boldsymbol{T}_j^b\,\boldsymbol{T}_k^c \ln\frac{-s_{ij}}{-s_{jk}} \ln\frac{-s_{jk}}{-s_{ki}} \ln\frac{-s_{ki}}{-s_{ij}}$$

- Non-trivial color structure only arises since his operators are not defined in a minimal scheme
- First derived by Mert Aybat, Dixon, Sterman '06 , confirming earlier conjecture Bern, Dixon, Kosower '04

Effective Theory Analysis

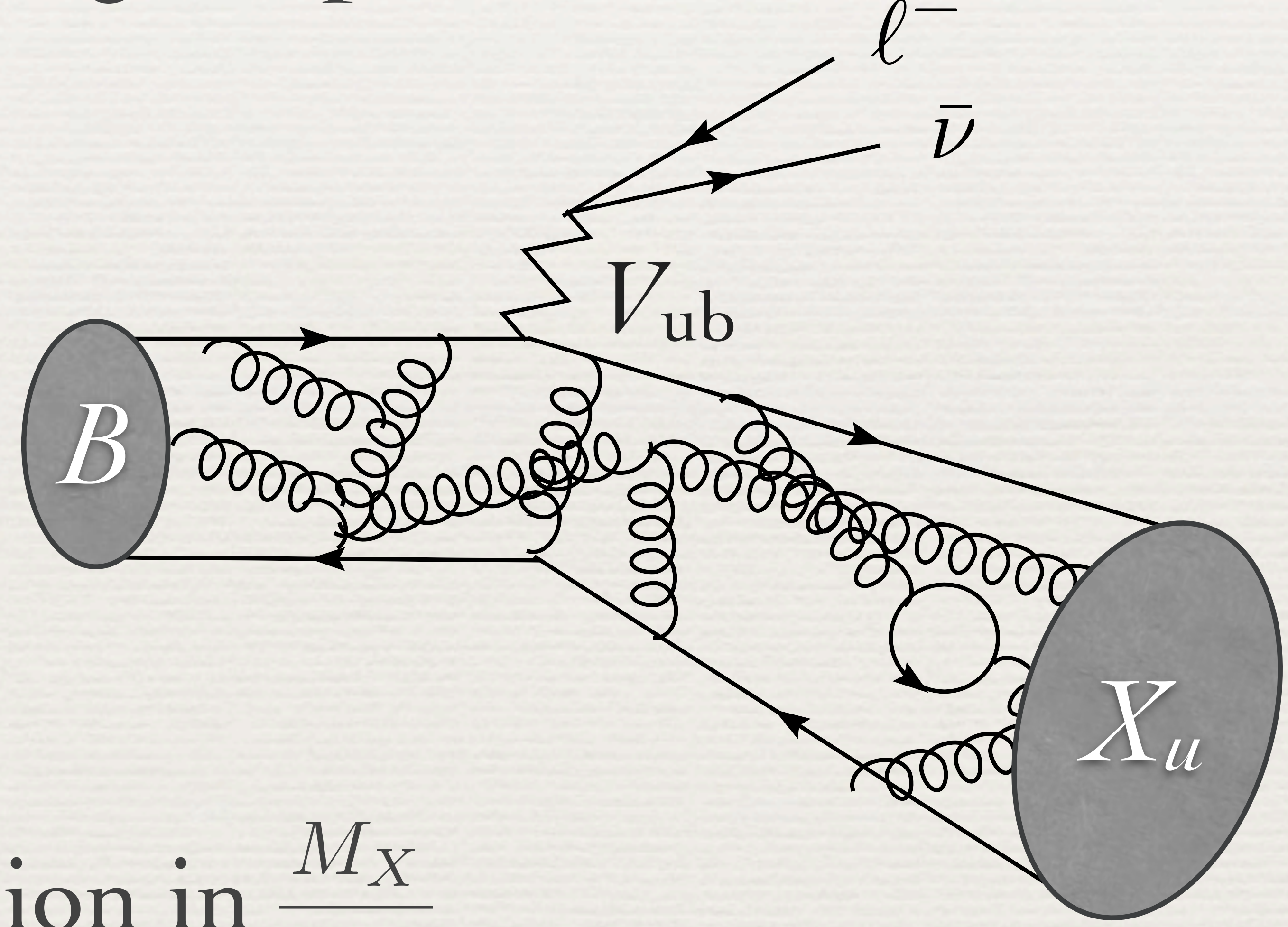
Misconception

- Conventional thinking is that UV and IR divergences are of totally different nature:
  - UV divergences absorbed into renormalization of parameters of theory; structure constrained by RG equations
  - IR divergences arise in unphysical calculations; cancel between virtual corrections and real emissions
- In fact, IR divergences can be mapped onto UV divergences of operators in effective field theory!

Soft-Collinear Effective Theory

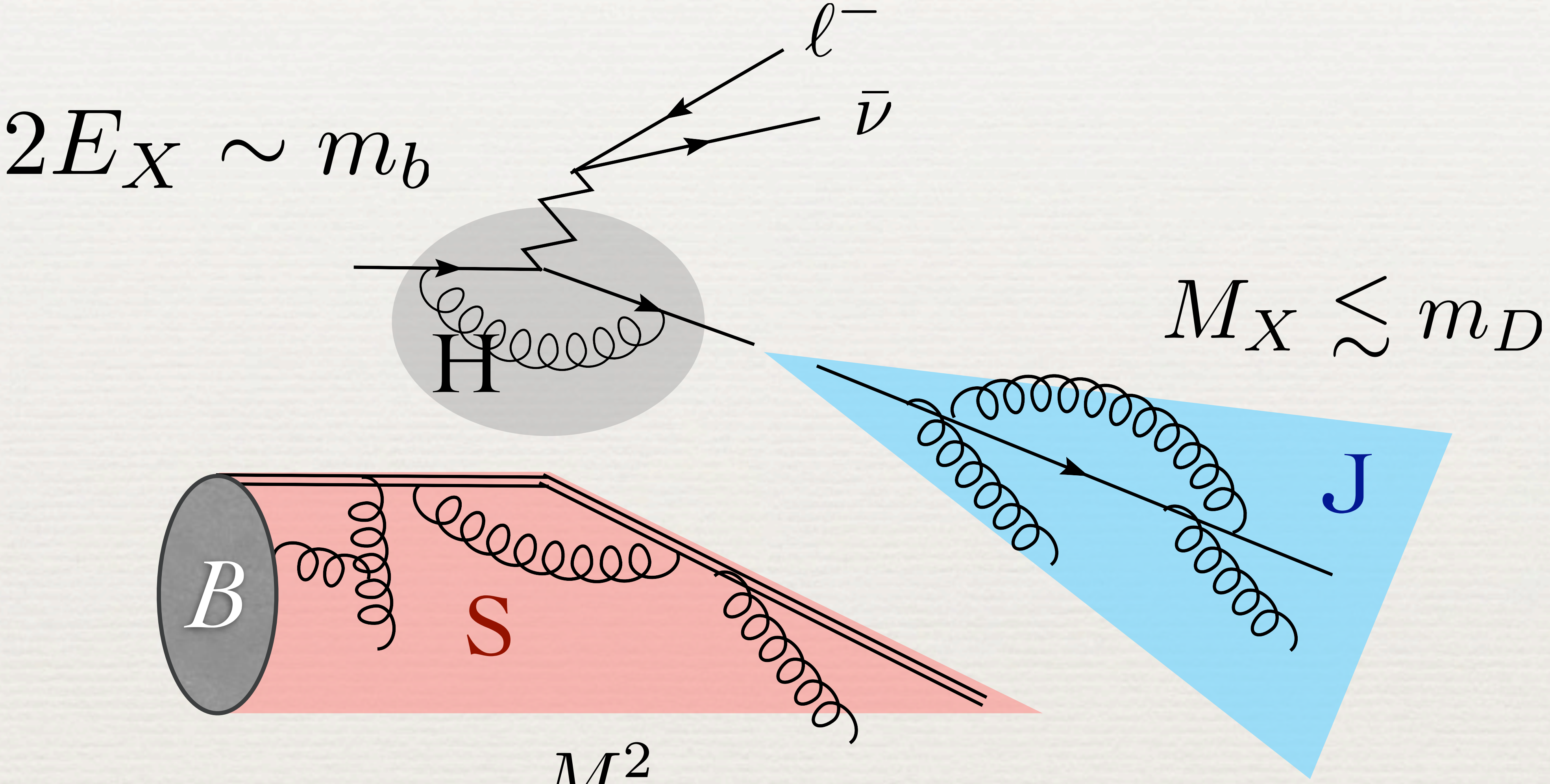
Bauer, Pirjol, Stewart et al. 2001, 2002; Beneke et al. 2002

- An effective theory for processes for processes with energetic particles.



- Expansion in $\frac{M_X}{2E_X}$
- Sudakov resummation $\alpha_s^n \ln^{2n}\left(\frac{M_X}{2\,E_X}\right)$

Soft-Collinear Factorization



$2E_X \sim m_b$

$M_X \lesssim m_D$

$\Lambda_S = \frac{M_X^2}{2E_X}$ nonperturbative!

$$d\Gamma = H \cdot J \otimes S$$

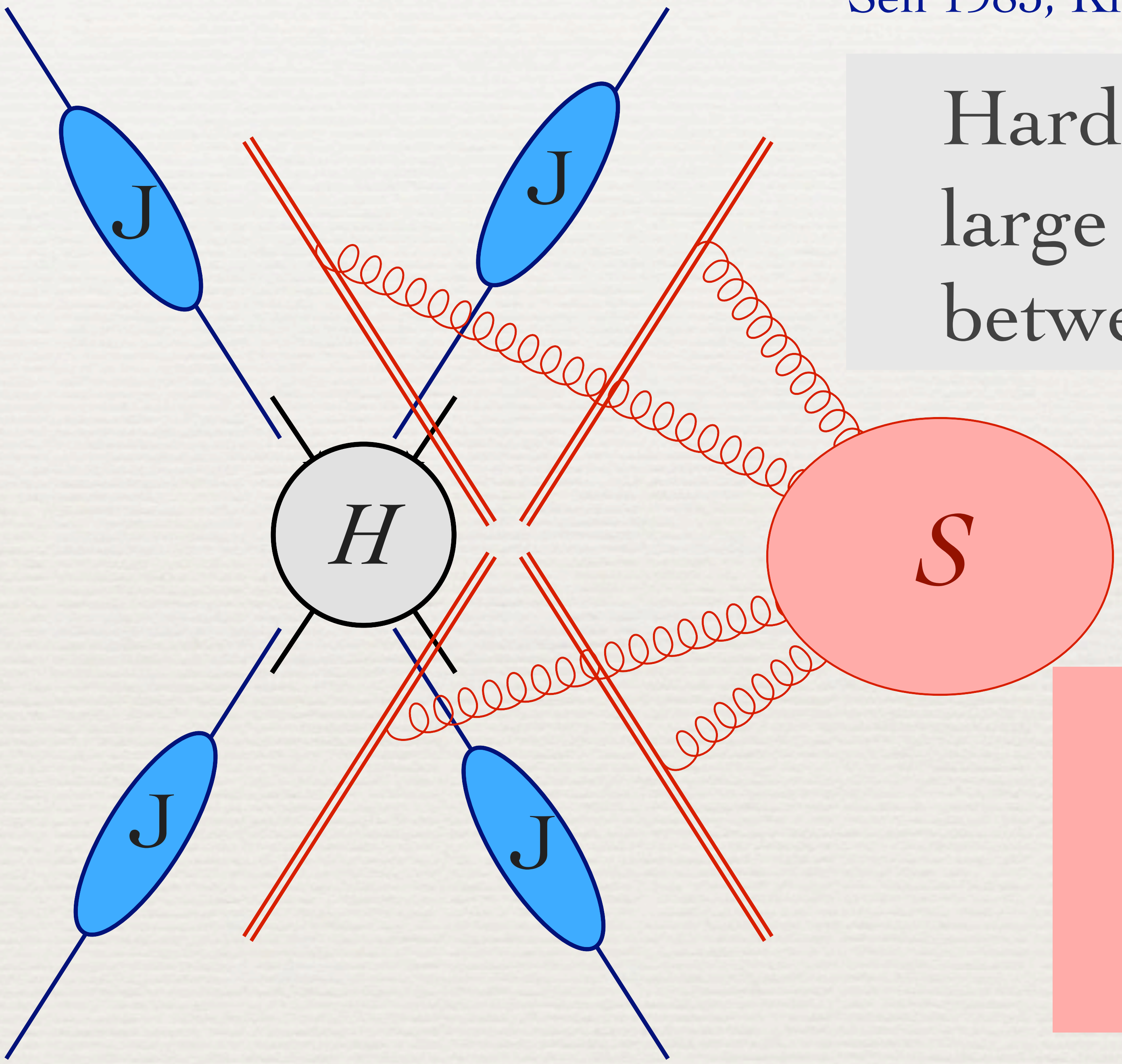
“Strategy of regions”

Beneke and Smirnov ’97

- A method to perform asymptotic expansions of loop integrals.
  - identify momentum regions of loop integration which lead to singularities
  - expand integrand in each region, integrate
  - boundary terms and non-singular regions give vanishing contribution in dim. reg.
- In SCET low-energy regions are represented by different fields. Hard contribution is absorbed into Wilson coefficient.

Soft-collinear factorization: *n* jet case

Sen 1983; Kidonakis, Oderda, Sterman 1998



Hard function *H* depends on large momentum transfers s_{ij} between jets

Soft function *S* depends on scales $\Lambda_{ij}^2 = \frac{M_i^2 M_j^2}{s_{ij}}$

Jet functions $J_i = J_i(M_i^2)$

SCET for *n*-jet processes

- *n* different types of collinear quark and gluon fields (→ jet functions J_i), interacting only via soft fields (soft function S)
  - operator definitions for J_i and S
- Hard contributions (Q ~ √s) are integrated out and absorbed into Wilson coefficients:

$$\mathcal{H}_n = \sum_i \mathcal{C}_{n,i}(\mu)\, O_{n,i}^{\rm ren}(\mu)$$

Bauer, Schwartz 2006

- Scale dependence controlled by RGE:

$$\frac{d}{d\ln\mu}\,|\mathcal{C}_n(\{\underline{p}\},\mu)\rangle = \boldsymbol{\Gamma}(\mu,\{\underline{p}\})\,|\mathcal{C}_n(\{\underline{p}\},\mu)\rangle$$

anomalous-dimension matrix

On-shell parton scattering amplitudes

- Hard functions C_n can be obtained by setting the jet masses to zero: jet and soft functions become scaleless, loop corrections vanish.
- One obtains:

renormalization factor
(minimal subtraction of IR poles)

$$|\mathcal{C}_n(\{\underline{p}\},\mu)\rangle = \lim_{\epsilon\to 0} \boldsymbol{Z}^{-1}(\epsilon,\{\underline{p}\},\mu)\,|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle$$

TB, Neubert 2009

where

$$\boldsymbol{\Gamma} = -\frac{d\ln \boldsymbol{Z}}{d\ln\mu}$$

- IR poles of scattering amplitudes mapped onto UV poles of n-jet SCET operators
- Multiplicative subtraction, controlled by RG

Factorization constraint on $\mathbf{\Gamma}$

- Operator matrix elements must evolve in the same way as hard matching coefficients, such that physical observables are scale independent
- Factorization of matrix element then implies (with $\Lambda_{ij}^2 = \frac{M_i^2 M_j^2}{s_{ij}}$):

trivial color structure

$$\boxed{\mathbf{\Gamma}(s_{ij}) = \mathbf{\Gamma}_s(\Lambda_{ij}^2) + \sum_i \Gamma_c^i(M_i^2)\,\mathbf{1}}$$

M_i dependence must cancel!

  - suggests logarithmic dependence on s_{ij} and M_i^2
  - $\mathbf{\Gamma}$ and $\mathbf{\Gamma}_s$ must have same color structure

Decoupling of soft interactions

- At leading power only a single component of the soft gluon field interacts with each collinear field.

$n_i \sim p_i$ light-like reference vector

$$\mathcal{L}_{c_i+s} = \bar{\chi}_i(x)\,\frac{\bar{\not{n}}_i}{2}\,n_i \cdot A_s(x_-)\,\chi_i(x)$$

collinear quark field in *i*th direction

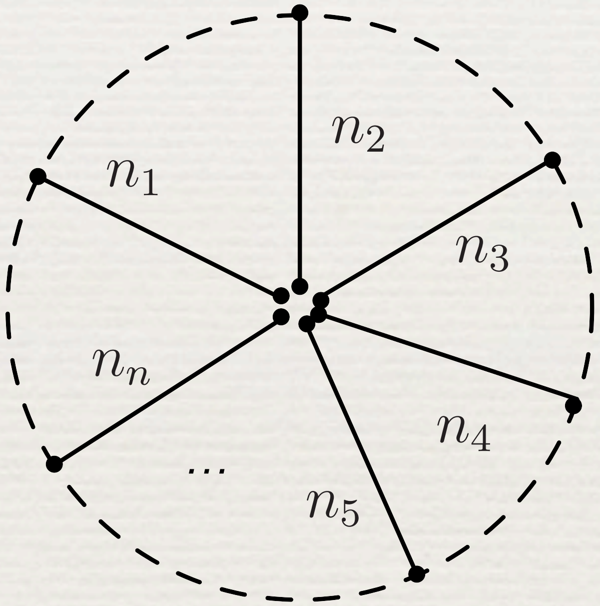
- Can decoupled by field redefinition

$$\chi_i(x) = S_i(x_-)\,\chi_i^{(0)}(x)$$

$$S_i(x) = \mathbf{P}\exp\left(ig\int_{-\infty}^{0} dt\,n_i \cdot A_s^a(x+tn_i)\,t^a\right)$$

Soft function *S*

- SCET decoupling transformation removes soft interactions from Lagrangian. The soft Wilson lines appear in the operators.



- For n-jet operator one gets:

$$\mathcal{S}(\{\underline{n}\},\mu) = \langle 0|\boldsymbol{S}_1(0)\dots\boldsymbol{S}_n(0)|0\rangle$$

$$\boldsymbol{S}_i = \mathbf{P}\exp\left[ig\int_{-\infty}^{0} dt\, n_i\cdot A_a(tn_i)\, T_i^a\right]$$

Perturbative results for *S*

- Our conjecture implies for the soft anomalous-dimension matrix:

$$\beta_{ij} = \ln \frac{-s_{ij}\,\mu^2}{M_i^2\,M_j^2}$$

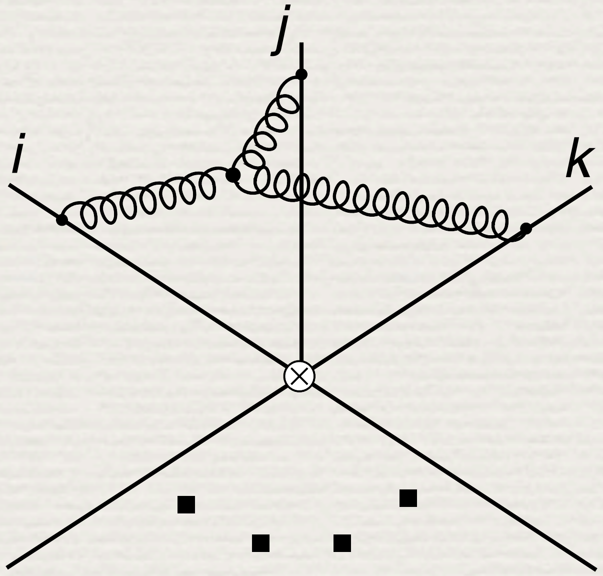
$$\mathbf{\Gamma}_s(\{\underline{\beta}\},\mu) = -\sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2}\,\gamma_{\text{cusp}}(\alpha_s)\,\beta_{ij} + \sum_i \gamma_s^i(\alpha_s)$$

- This form was obtained at two loops by showing that diagrams connecting three parton legs vanish

  Mert Aybat, Dixon, Sterman 2006

- Also holds for three-loop fermionic contributions

  Dixon 2009



Analysis of Sterman and Tejeda-Yeomans '03

- Based on factorization

$$|\mathcal{M}_n\rangle = \prod_i \underset{\substack{\uparrow\\ \text{color-diagonal}}}{J_i(\alpha_s,\epsilon)}\, \underset{\substack{\uparrow\\ \text{eikonal}}}{\boldsymbol{S}(\alpha_s,\epsilon)}\, \underset{\substack{\uparrow\\ \text{finite}}}{|h_n(\alpha_s)\rangle}$$

- Define jet-function as square root of form factor $J_i(\alpha_s,\epsilon) = [F(Q^2)]^{1/2}$
- Structure of IR divergences governed by $\boldsymbol{S}$
- Same physical picture, but rather different definition of hard, jet and soft functions
  - In SCET $|\mathcal{M}_n\rangle$ is purely hard, since it only depends on hard scales.

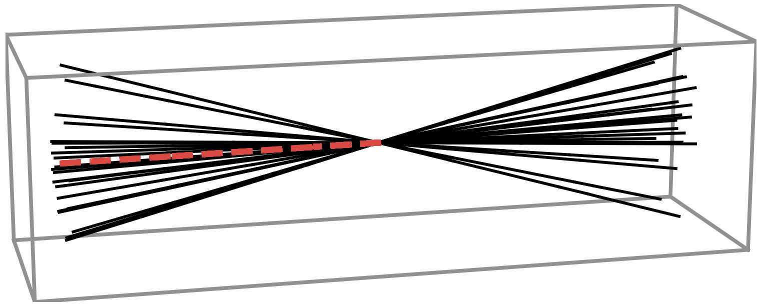
Towards higher-log resummations for *n*-jet processes

Sudakov resummation with SCET

- Many collider physics applications of SCET in the past few years. Resummations up to N^{3}LL, however only for two jet observables, e.g.
  - Drell-Yan rapidity dist. TB, Neubert, Xu '07
  - inclusive Higgs production Idilbi, Ji, Ma and Yuan '06 ; Ahrens, TB, Neubert, Yang '08
  - thrust distribution in e^+e^- TB, Schwartz '08
- Our result for anomalous dimension $\boldsymbol{\Gamma}$ allows us to perform higher-log resummations also for n-jet processes

2-jet example: thrust T

$$T = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p}_i \cdot \mathbf{n}|}{\sum_i |\mathbf{p}_i|}$$



$$1 - T \approx \frac{M_1^2 + M_2^2}{Q^2}$$

- Prediction for event-shape variable thrust dominated by perturbative uncertainty. NLO Ellis et al. '81, NNLO corrections Gehrmann et al. '07.
  - Traditional methods allowed resummation to NLL Catani et al. '93 but not beyond.
- Using factorization theorem in SCET we were able to derive NNNLL resummed distribution TB and Schwartz, '08.
  - Need only existing perturbative input. Analytic result, no unphysical Landau-pole singularities. Match to NNLO.
  - Observe dramatic improvement of convergence.

Higgs production $pp \to H{+}X$

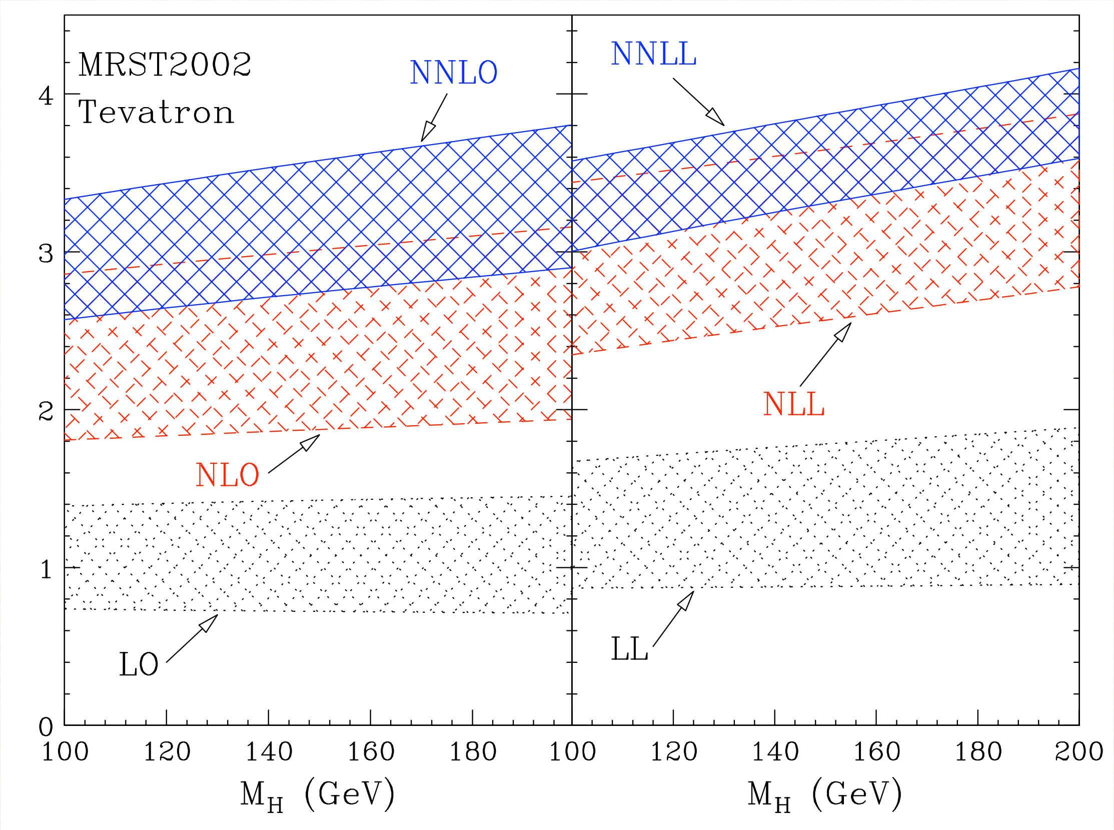
- Factorization theorem for partonic cross section near threshold $z = m_H^2/\hat{s} \to 1$

$$\sigma_{\rm part} = C_t(m_t^2,\mu^2)\,H(m_H^2,\mu^2)\,S(m_H^2(1-z)^2,\mu^2)$$

- Can solve RG equations for the different parts: this resums log's of scale ratios.
  - equivalent to soft-gluon resummation
- Soft scale is set dynamically via the fall-off of the PDF. For $m_{\rm H}$= 120 GeV,

$$\sigma_{\rm had} \propto \int_0^1 dz\, z^{1.5} \sigma_{\rm part}(z)$$

weight function
not strongly peaked
near z=1



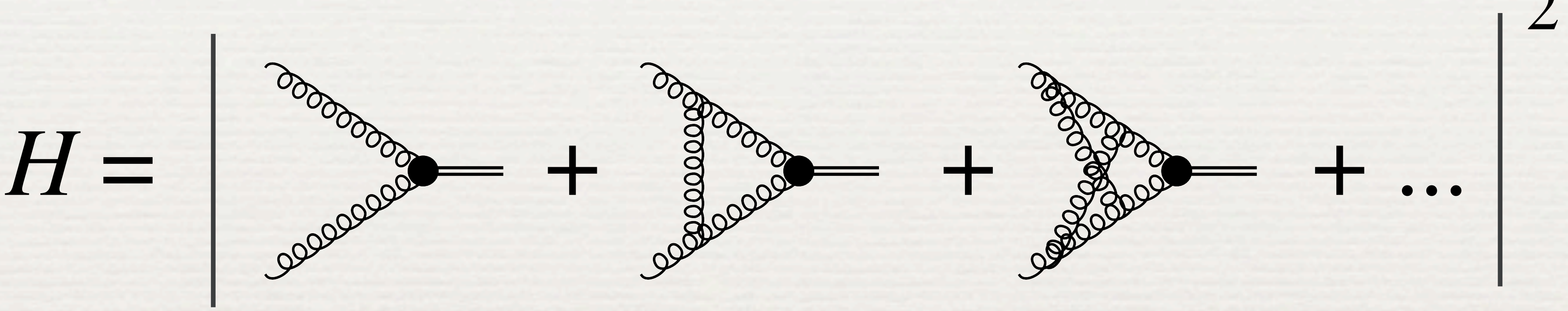
Catani, de Florian, Grazzini, Nason '03

- Soft scale is ~ $m_H/2$, not much lower than hard scale. No large soft logarithms.
    - however, the threshold region is numerically large, gives ~ 90% of NLO and NNLO correction
- Even after resummation of log's, higher order corrections are very large.

Origin of the large corrections

Ahrens, TB, Neubert, Yang '08; → talk by Li Lin Yang at Pheno

- Hard function gets large higher order corrections



$$H(m_H^2, \mu^2 = m_H^2) \quad = \quad 1 + 5.50\alpha_s(m_H^2) + 17.24\alpha_s^2(m_H^2) + \ldots$$
$$= \quad 1 + 0.623 + 0.221 + \ldots$$

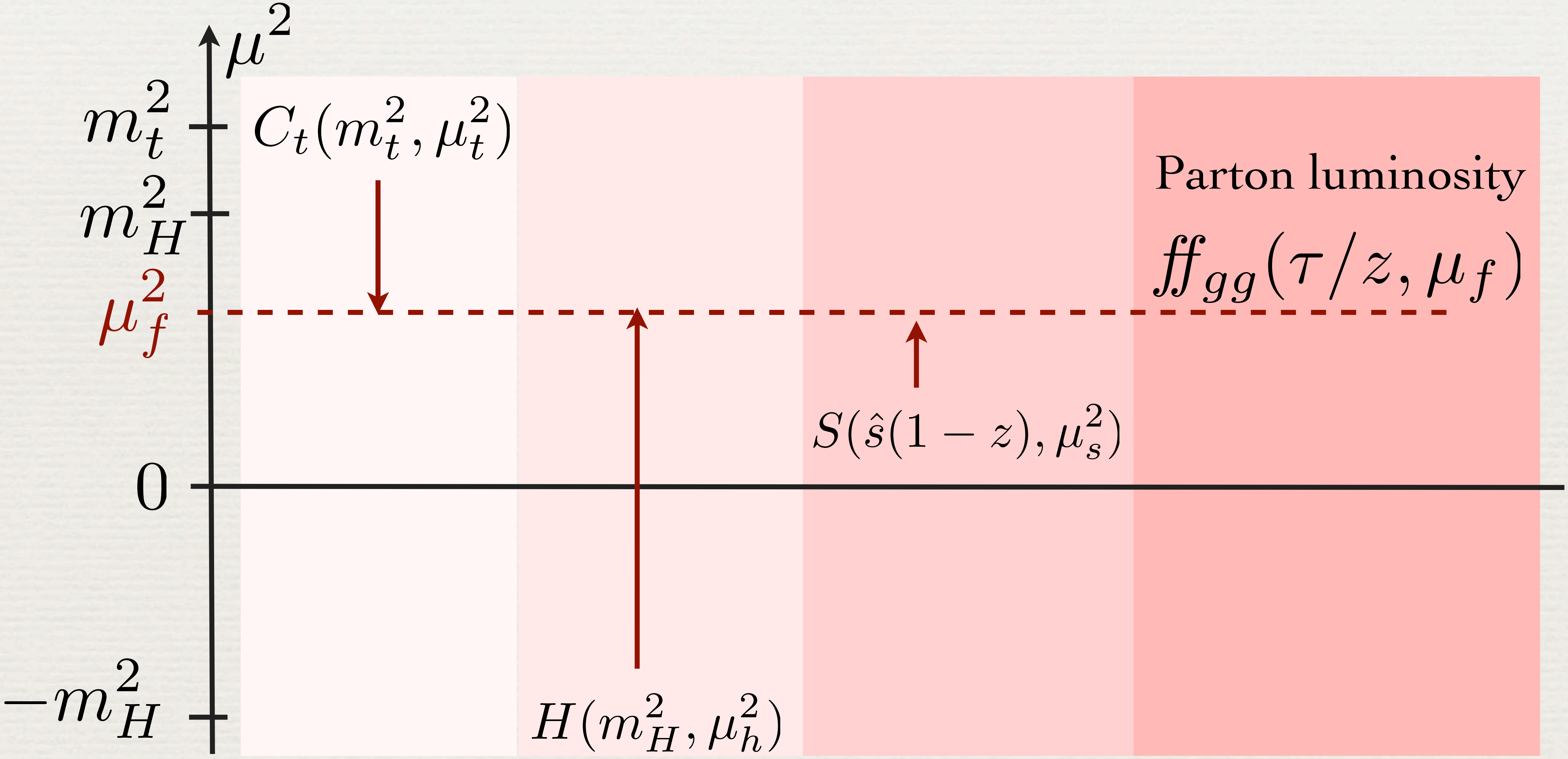
- The space-like form factor has well behaved expansion:

$$H(m_H^2, \mu^2 = -m_H^2) \quad = \quad 1 - 0.15 - 0.0012 + \ldots$$

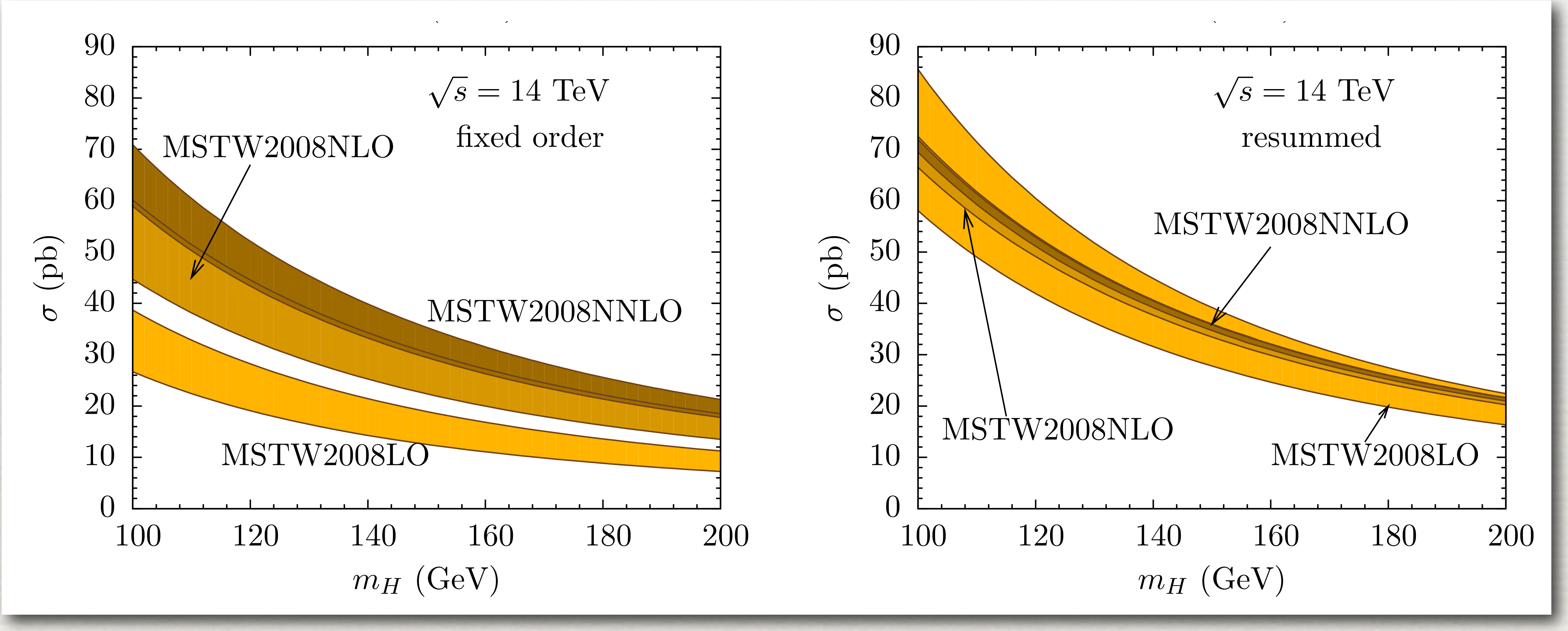
  - use RG to evolve back to $\mu^2 = +m_H^2$

Resummation by RG evolution

- Evaluate each part at its characteristic scale, evolve to common scale:



Numerical results

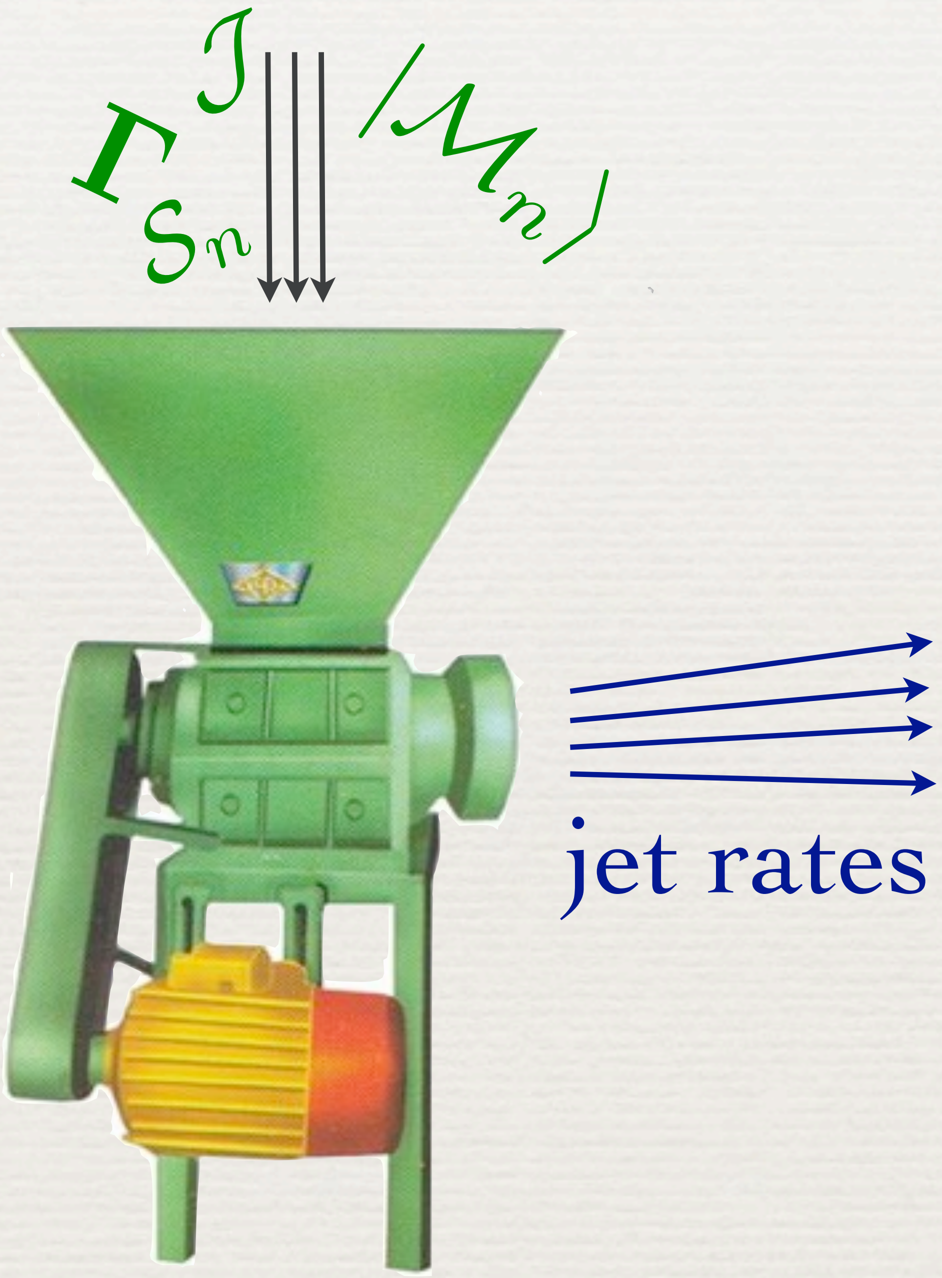


- Includes soft-gluon resummation, but the main effect arises from scale setting $\mu^2 = -m_H^2$ in hard function.
  - RG improved NNLO result is 8% larger than fixed order (13% at Tevatron).

N^kLL for *n*-jet processes

- The necessary ingredients are
  - **hard functions**: from fixed-order results for on-shell amplitudes. New unitarity methods allow calculation of one-loop amplitudes with many legs (→ NNLL resummation)
  - **jet function:** imaginary part of two-point function, inclusive jet function is known to two loops.
  - **soft function:** matrix element of Wilson lines, one-loop calculation is comparatively simple.
- Then resum log's of different scales using RG evolution.

Automatization



- in the longer term, this will hopefully lead to automated higher-log resummations for jet rates
- goes beyond parton showers, which are only accurate at LL, even after matching
- predicts jets, not individual partons

Conclusion

- IR divergences of scattering amplitudes in gauge theories can be absorbed into multiplicative **Z**-factor, derived from anomalous dimension $\boldsymbol{\Gamma}$ of operators in SCET.
- Form of $\boldsymbol{\Gamma}$ is severely constrained from non-abelian exponentiation, soft-collinear factorization and collinear limits → Matthias' talk
  - we conjecture $\boldsymbol{\Gamma}$ to have only dipole color-correlations to all orders in PT.
- Are on track to perform higher-log resummation for n-jet processes at LHC using RG evolution SCET.