Precision Gravity from Effective Field Theory

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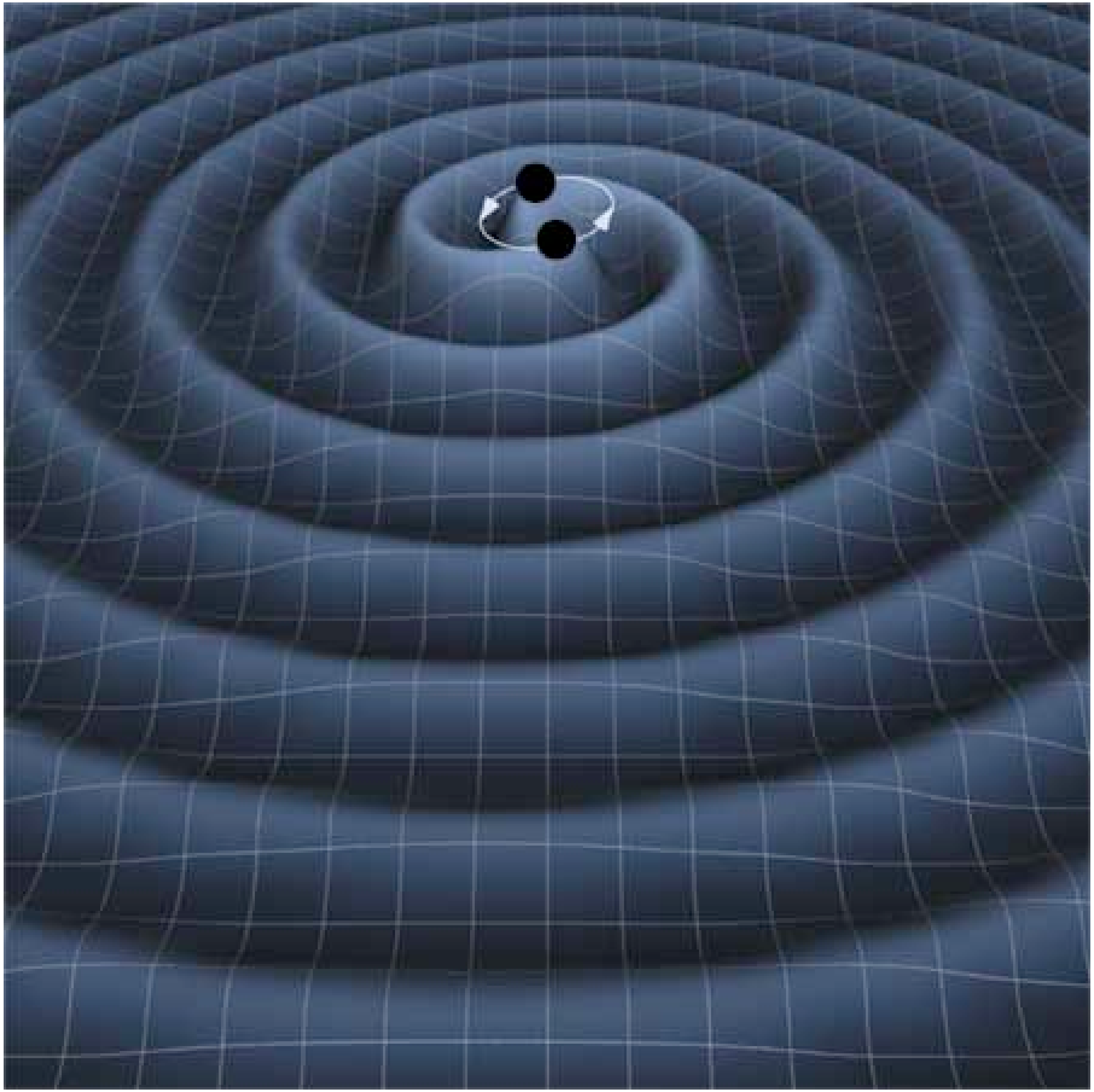
based on (w/ I. Rothstein):
hep-th/0409156
hep-th/0511133
hep-th/060216, hep-th/0605238
(review: hep-ph/0701129)
+ in progress with A. Ross, I . Rothstein

In this talk I will discuss the dynamics of gravitationally bound systems composed of black hole (BH) or neutron star (NS) constituents:



There are a number of reasons why these systems are physically well motivated:

1. Realized astrophysically (e.g., the Hulse-Taylor binary pulsar, 1972)

2. Strong emitters of gravitational radiation, so are relevant to the experimental program in gravitational waves (LIGO/VIRGO, LISA)

3. Their dynamics is characterized by a hierarchy of length scales.

$\Longrightarrow$ Natural description in terms of **Effective Field Theories**

(WG+ I.Z. Rothstein, PRD 2005)

New applications of Feynman integral methods

Binary Inspirals and LIGO

An interesting class of signals for gravitational wave detectors (eg, LIGO/LISA) consists of radiation induced inspiral of compact binary systems (BH or NS constituents). Why?

1. Such systems are strong emitters of gravitational radiation:

$$h \sim \frac{1}{R} G_N E_{int} \sim 10^{-19} \frac{1}{R(\mathrm{Mpc})} \frac{m}{m_\odot}$$

giving values in the LIGO range $h = \Delta L/L \sim 10^{-21} - 10^{-22}$ for, eg, solar mass NS/NS at $R \sim 3000\mathrm{Mpc}$

2. Many expected inspiral events per year for upgraded LIGO:

	NS/NS	NS/BH	BH/BH in field	BH/BH in clusters
$\mathcal{R}_{\rm gal}$, yr^{-1}	10^{-6}–5×10^{-4}	$\lesssim 10^{-7}$–10^{-4}	$\lesssim 10^{-7}$–10^{-5}	$\sim 10^{-6}$–10^{-5}
$D_{\rm I}$	20 Mpc	43 Mpc	100	100
$\mathcal{R}_{\rm I}$, yr^{-1}	$3 \times 10^{-4} - 0.3$	$\lesssim 4 \times 10^{-4} - 0.6$	$\lesssim 4 \times 10^{-3} - 0.6$	$\sim 0.04 - 0.6$
$D_{\rm WB}$	300 Mpc	650 Mpc	$z = 0.4$	$z = 0.4$
$\mathcal{R}_{\rm WB}$, yr^{-1}	$1 - 800$	$\lesssim 1 - 1500$	$\lesssim 30 - 4000$	$\sim 300 - 4000$

(from Cutler+Thorne, gr-qc/0204090)

3. Signal is long duration: For binary in close but non-relativistic orbit, can use virial thm. to get estimate of orbital dynamics:

$$v^2 \sim \frac{G_M m}{r} \equiv \frac{r_s}{2r}$$

e.g LIGO can detect signals in a frequency band $10 \text{ Hz} < \nu < 1 \text{ kHz}$
This correspond to orbital parameters

$$r(10 \text{ Hz}) \sim 300 \text{ km} \left(\frac{m}{m_\odot}\right)^{1/3} \rightarrow r(1 \text{ kHz}) \sim 14 \text{ km} \left(\frac{m}{m_\odot}\right)^{1/3}$$

(for comparison, $r_s \sim 1 \text{ km}$ for $m \sim m_\odot$)

$$v(10 \text{ Hz}) \sim 0.06 \left(\frac{m}{m_\odot}\right)^{1/3} \rightarrow v(1 \text{ kHz}) \sim 0.3 \left(\frac{m}{m_\odot}\right)^{1/3}$$

(note: $m_{NS} \sim m_\odot, m_{BH} \sim 10 m_\odot$).

In this regime, the energy loss of the binary to GW's is approx. given by the quadrupole rad. formula:

$$\frac{d}{dt}\left(-\frac{1}{2}mv^2\right) = -\frac{32}{5}G_N^{-1}v^{10}$$

Solving this gives:

$$\Delta t \sim \frac{5}{512}\left(\frac{1}{v_i^8} - \frac{1}{v_f^8}\right) \sim 5 \text{ min.} \left(\frac{m}{m_\odot}\right)^{-8/3}$$

for the duration of the inspiral event in the LIGO band, and

$$N \sim \int_{t_i}^{t_f} \omega(t)dt = \frac{1}{32}\left(\frac{1}{v_i^5} - \frac{1}{v_f^5}\right) \sim 4 \times 10^4 \left(\frac{m}{m_\odot}\right)^{-5/3} \text{radians}$$

for the number of orbital cycles spent in the detector band.

Large number of orbital cycles in LIGO band

LIGO is sensitive to at least $(v/c)^6$ corrections beyond Newtonian gravity for binary dynamics. (Cutler et. al. astro-ph/9208005)

(even higher order corrections may also important for BH/BH + matching to numerical GR)

What can be learned:

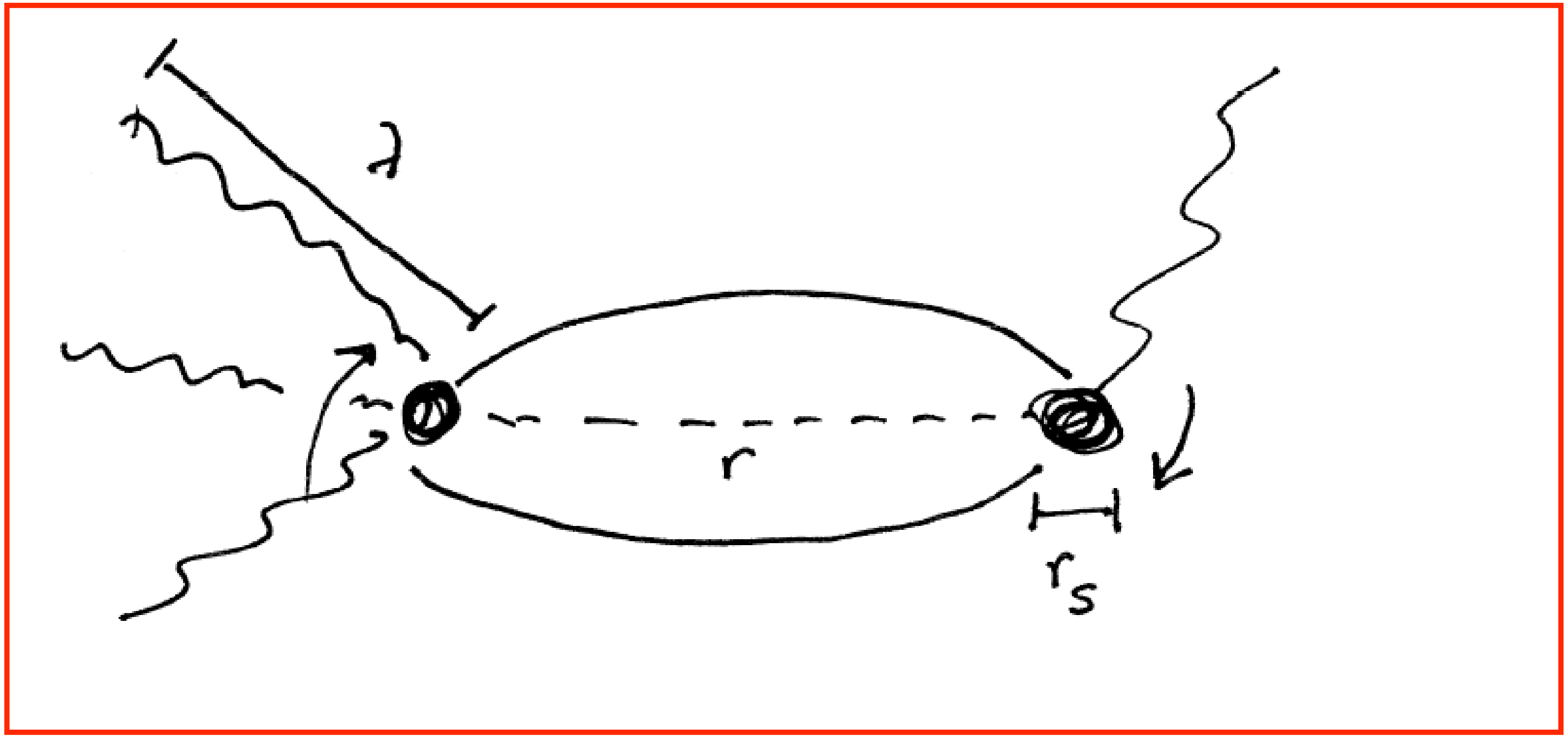
1. Accurate parameter extraction (masses, spins, distances) for compact binaries out to $R \sim 1000\mathrm{Mpc}$.

2. Stringent tests of (classical) General Relativity.

"Precision gravity"

3. Structure of black holes or neutron stars? (eg., dynamics of BH horizons?)

Non-relativistic binary problem is also interesting from the point of view of field theory, as it is a problem with many different length scales. E.g, for binary BH:



$r_s =$ Black hole radius $\quad r =$ Orbital radius

$\lambda =$ Wavelength of grav. radiation

Scales are correlated: $r \sim r_s/v^2$ (virial thm.) $\lambda \sim r/v$

$v \ll 1$ Typical three-velocity

Because scales are correlated, a single expansion parameter controls qualitatively different physical effects....

This motivates an Effective Field Theory formulation of the binary problem. For example, in the EFT language:

1. Only one scale appears at a time in the perturbative expansion. This simplifies the (Feynman) integrals.

2. Can regularize UV (multipole) and IR (Coulomb) divergences in dimensional regularization, by the usual methods.

3. Use RG to sum logs of scale ratios $= \ln v$

4. Feynman vertices can be automatized in a form suitable for computer algebra (Mathematica).

(Alternative to "post-Newtonian" expansion of Blanchet, Damour, Schafer,... (EU) and Will et. al. (US))

Binary stars as an EFT problem

Binary problem involves a hierarchy of scales, $r_s \ll r \ll r/v$

Simplify by integrating out one at a time

Extended objects (BH/NS) + GR (the "full theory")

match $\qquad \dfrac{1}{\mu} = r_s$

Relativistic point particle + GR

$\downarrow$

2-body NR problem

match $\qquad \dfrac{1}{\mu} = r$

Composite object + radiation gravitons (NRGR)

$\downarrow$ non-trivial RG running

$\dfrac{1}{\mu} = \dfrac{r}{v}$

Starting point is a theory of point particles coupled to metric tensor $g_{\mu\nu}(x)$

$$S = S_{EH} + S_{pp}$$

Gravitational dynamics:

$$S_{EH} = -2m_{Pl}^2 \int d^4x \sqrt{g} R(x) \qquad (m_{Pl}^2 = 1/(32\pi G_N))$$

BH/NS sources: Write most general diffeomorphism invariant Lagrangian

$$S_{pp} = -m \int d\tau + c \int d\tau R_{\mu\nu\alpha\beta}^2 + \cdots$$

geodesic (test particle) motion

finite size corrections

$$d\tau^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

Spin dofs: (see R. Porto + I. Rothstein, 2006-2008)

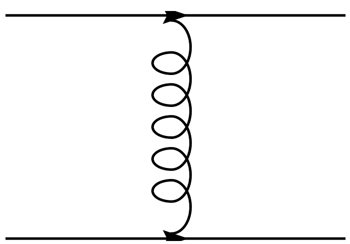
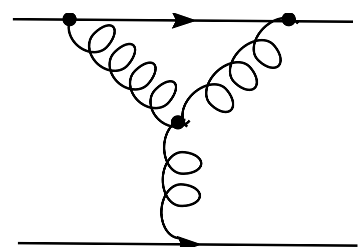
Calculating Observables:

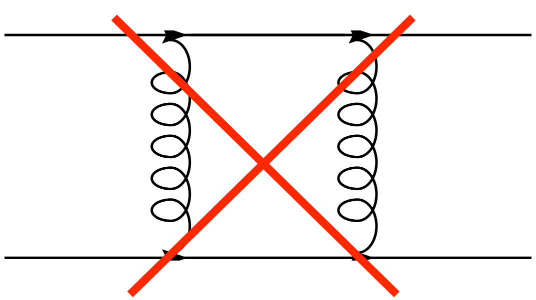
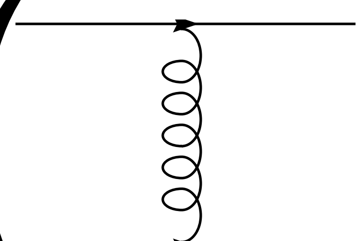
In principle, all observables can be read off the "gravitational Wilson loop"

$$\exp[iS_{eff}(x_a)] = \int \mathcal{D}h_{\mu\nu}\,\exp[iS_{EH} + iS_{pp}]. \qquad (g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu})$$

holding the particle coordinates $x_a^\mu(\tau)$ fixed. Diagrammatically

$iS_{eff}(x_a) =$ Feynman diagrams that remain connected if particle lines are removed (no graviton loops)

$iS_{eff}(x_a) =$  $+$  $+\cdots$

 $= \Big($  $\Big)^2$

(no propagators for particle lines)

(Donoghue, gr-qc/9512024)

These diagrams are computed w/ conventional Feynman rules from $S_{EH} + S_{pp}$

$S_{eff}(x_a)$ generates all relevant observables:

$$\mathrm{Re} S_{eff}[x_a] \longrightarrow$$ classical e.o.m.'s for x_a^μ

$$\frac{1}{T}\mathrm{Im} S_{eff}[x_a] = \frac{1}{2}\int dE d\Omega \frac{d^2\Gamma}{dE d\Omega}, \longrightarrow$$ gives the differential decay rate for graviton emission

(for fixed particle paths $\{x_a^\mu\}$ and a large time $T \to \infty$)

From the rate obtain the radiated energy in gravitational waves

$$P = \int dE d\Omega \left[E \frac{d^2\Gamma}{dE d\Omega}\right]$$

By conservation of energy this is equal to mechanical energy loss

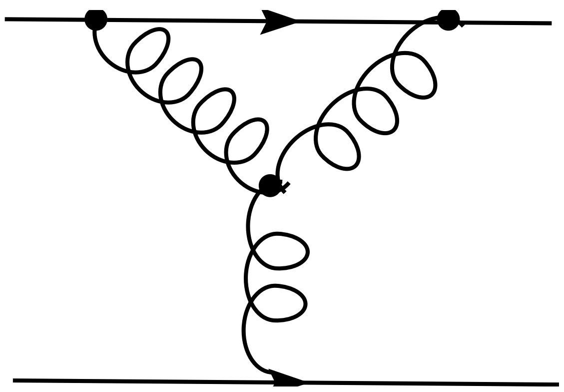
$$\frac{dE}{dt} = -P \longrightarrow \omega(t) \qquad \phi(t) = \int^t d\tau \omega(t)$$

GW freq.

GW phase

The presence of particle sources breaks spatial translations. Thus spatial momentum is not conserved and must be integrated over in Feynman diagrams.

Even though calculations involve diagrams of tree topology only, interesting Feynman integral structures arise in calculations. E.g.

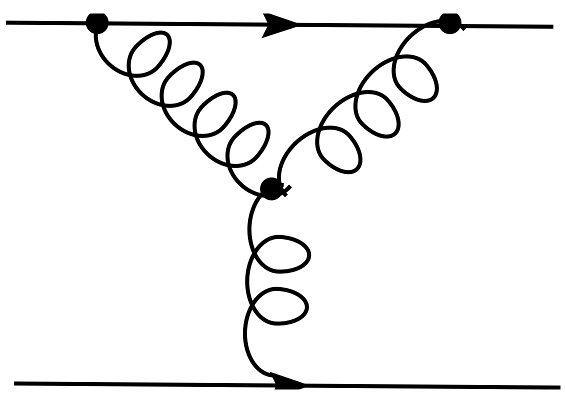


is a Feynman integral that has contributions from two regions of momenta

Potential: $(k^0 \sim v/r, \mathbf{k} \sim 1/r)$

Radiation: $(k^0 \sim v/r, \mathbf{k} \sim v/r)$

Thus the pt. particle + gravity theory used to generate the Feynman rules contains scales that have not been properly disentangled. Perturbation theory does not scale homogeneously with the expansion parameter $v \ll 1$



$$\sim v^?$$

The solution to this problem is well known from QCD:

Construct an EFT in which fluctuations in each momentum region correspond to a separate field (NRQCD, SCET,...)

Split graviton field into:

$$h_{\mu\nu}(x) = \bar{h}_{\mu\nu}(x) + H_{\mu\nu}(x),$$

radiation graviton

potential graviton

$$\partial_\alpha \bar{h}_{\mu\nu} \sim \frac{v}{r}\bar{h}_{\mu\nu}.$$

$$H_{\mu\nu}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} H_{\mathbf{k}\mu\nu}(x^0)$$

$$\partial_0 H_{\mathbf{k}\mu\nu} \sim \frac{v}{r} H_{\mathbf{k}\mu\nu}$$

In addition, need to multipole expand (as in NRQCD) the couplings of $\bar{h}_{\mu\nu}$ to either particles or potential modes. This gives a Lagrangian whose Feynman rules scale as definite powers of velocity:

$$x^\mu \sim r/v \quad \mathbf{k} \sim 1/r \qquad m/m_{Pl} \sim \sqrt{vL} \qquad (L = mvr \gg 1)$$

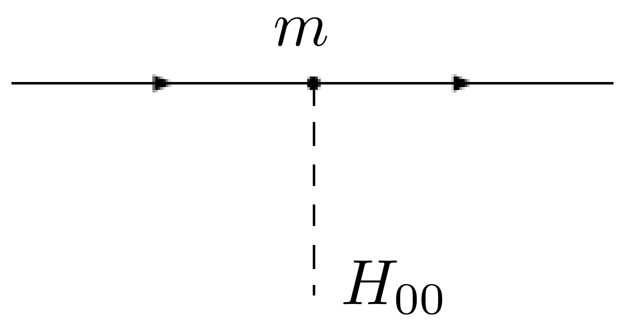
$$H_{\mathbf{k}\mu\nu}/m_{Pl} \sim v^2/\sqrt{L} \qquad \bar{h}_{\mu\nu}/m_{Pl} \sim v^{5/2}/\sqrt{L}$$

Loop counting

$(n \le 1, k \ge 0)$ $\boxed{\text{Any Feynman diagram scales as } L^n v^k}$

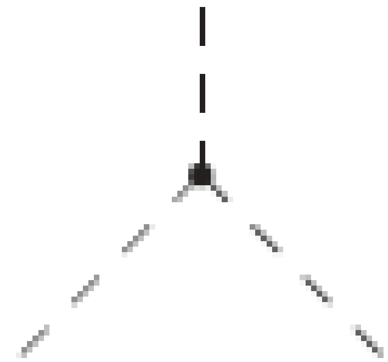
Examples:

Pt. particle Newton potential interaction:



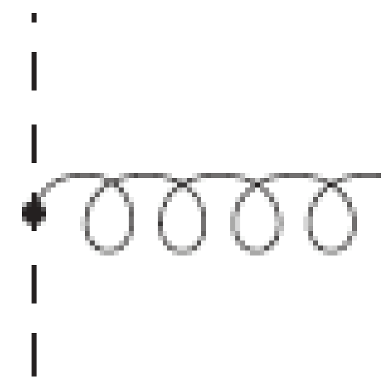
$$\sim \sqrt{L}$$

Potential 3-graviton vertex:



$$\sim \frac{v^2}{\sqrt{L}}$$

Radiation-potential interaction:



$$\sim \frac{v^{5/2}}{\sqrt{L}}$$

Integrate out potential modes to get effective Lagrangian for radiation modes:

$$\Gamma[\bar{h}] = \underbrace{\Gamma_0}_{\mathcal{O}(\bar{h}^0)} + \underbrace{\Gamma_1}_{\mathcal{O}(\bar{h}^1)} + \cdots$$

$$\Gamma_0 = \int dt L[\mathbf{x}_a] =$$ many-particle Lagrangian, Feynman graphs w/ no ext. $\bar{h}_{\mu\nu}$

$$\Gamma_1 = -\frac{1}{2m_{Pl}} \int d^4x T^{\mu\nu}(x) \bar{h}_{\mu\nu}(x)$$

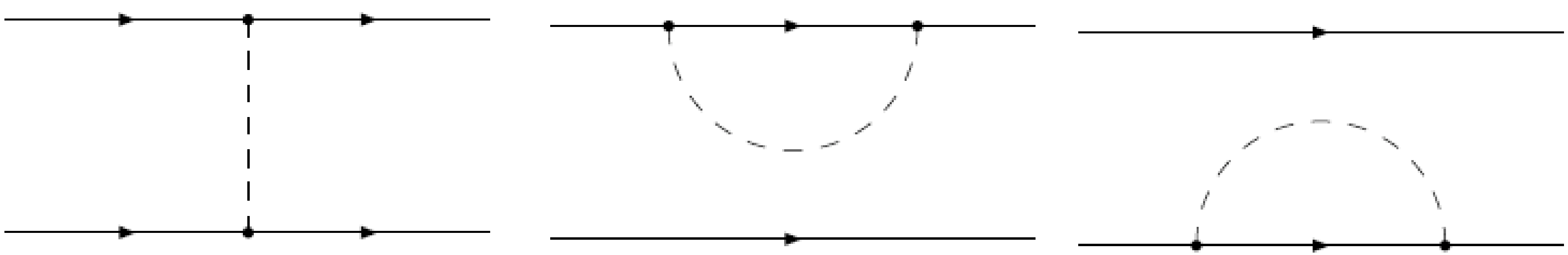
from graphs with one ext. $\bar{h}_{\mu\nu}$

$$T^{\mu\nu}(x) =$$ grav. energy-mom. "pseudo-tensor"

$$\partial_\mu T^{\mu\nu} = 0$$ (Ward id.)

Zero graviton sector: 2-body potentials

LO: $\mathcal{O}(v^0)$



(a) (b) (c)

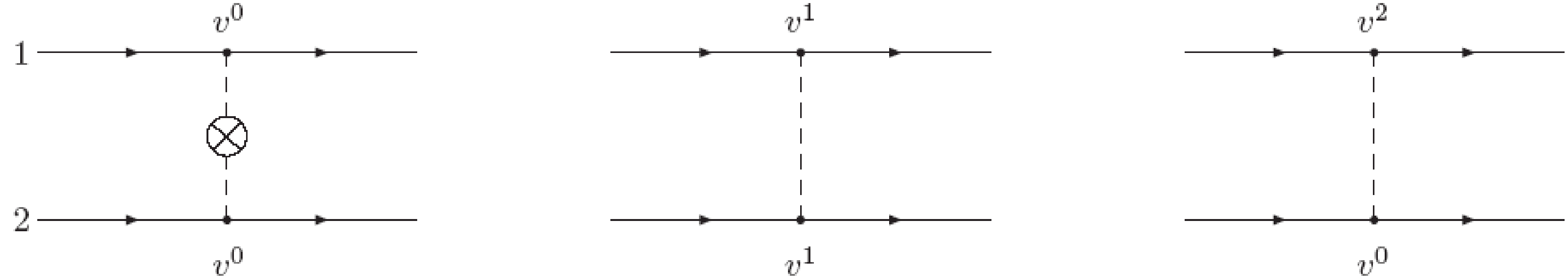
(b),(c): Calculating in dim. reg. $\sim \int \frac{d^{3-\epsilon}\mathbf{k}}{(2\pi)^{3-\epsilon}} \frac{1}{\mathbf{k}^2} = 0$

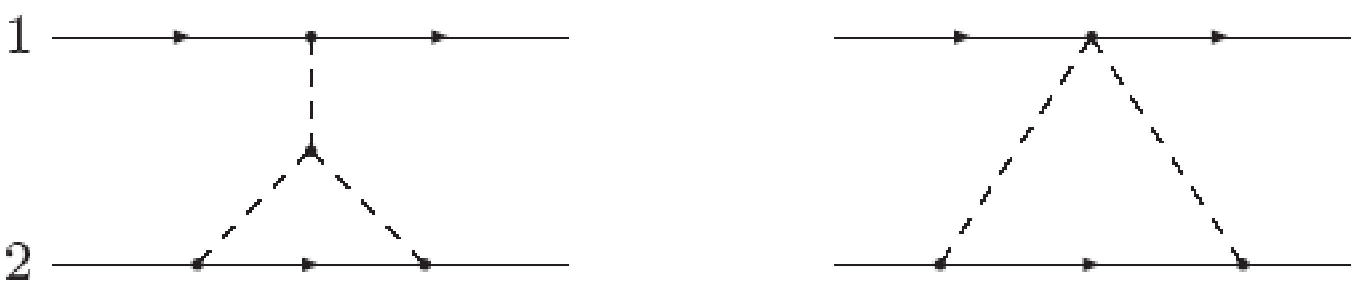
(a) $\left(-\frac{im_1}{2m_{Pl}}\right)\left(-\frac{im_2}{2m_{Pl}}\right)\int dt_1 dt_2 \langle H_{00}(x_1) H_{00}(x_2)\rangle \quad = \frac{im_1 m_2}{32\pi m_{Pl}^2}\int dt \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}$

$$L = \frac{1}{2}\sum_a m_a \vec{v}_a^2 + \frac{G_N m_1 m_2}{r}$$

(Newton)

NLO: $\mathcal{O}(v^2)$





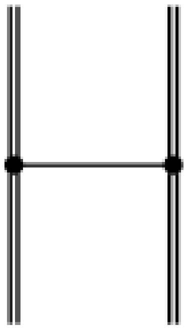
$+(1 \leftrightarrow 2)$

$$L_{EIH} = \frac{1}{8}\sum_a m_a \vec{v}_a^4 + \frac{G_N m_1 m_2}{2r}\left[3(\vec{v}_1^2 + \vec{v}_2^2) - 7\vec{v}_1\cdot\vec{v}_2 - (\vec{v}_1\cdot n)(\vec{v}_1\cdot n)\right]$$

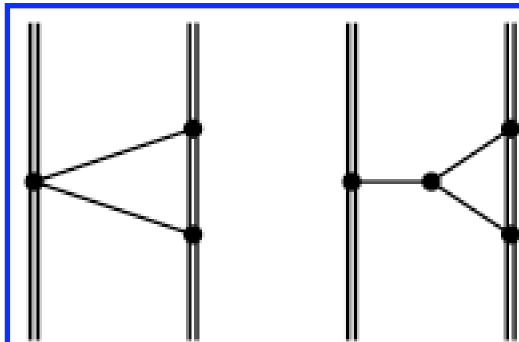
$$-\frac{G_N^2 m_1 m_2}{2r^2}$$

Einstein-Infeld-Hoffman (1938)

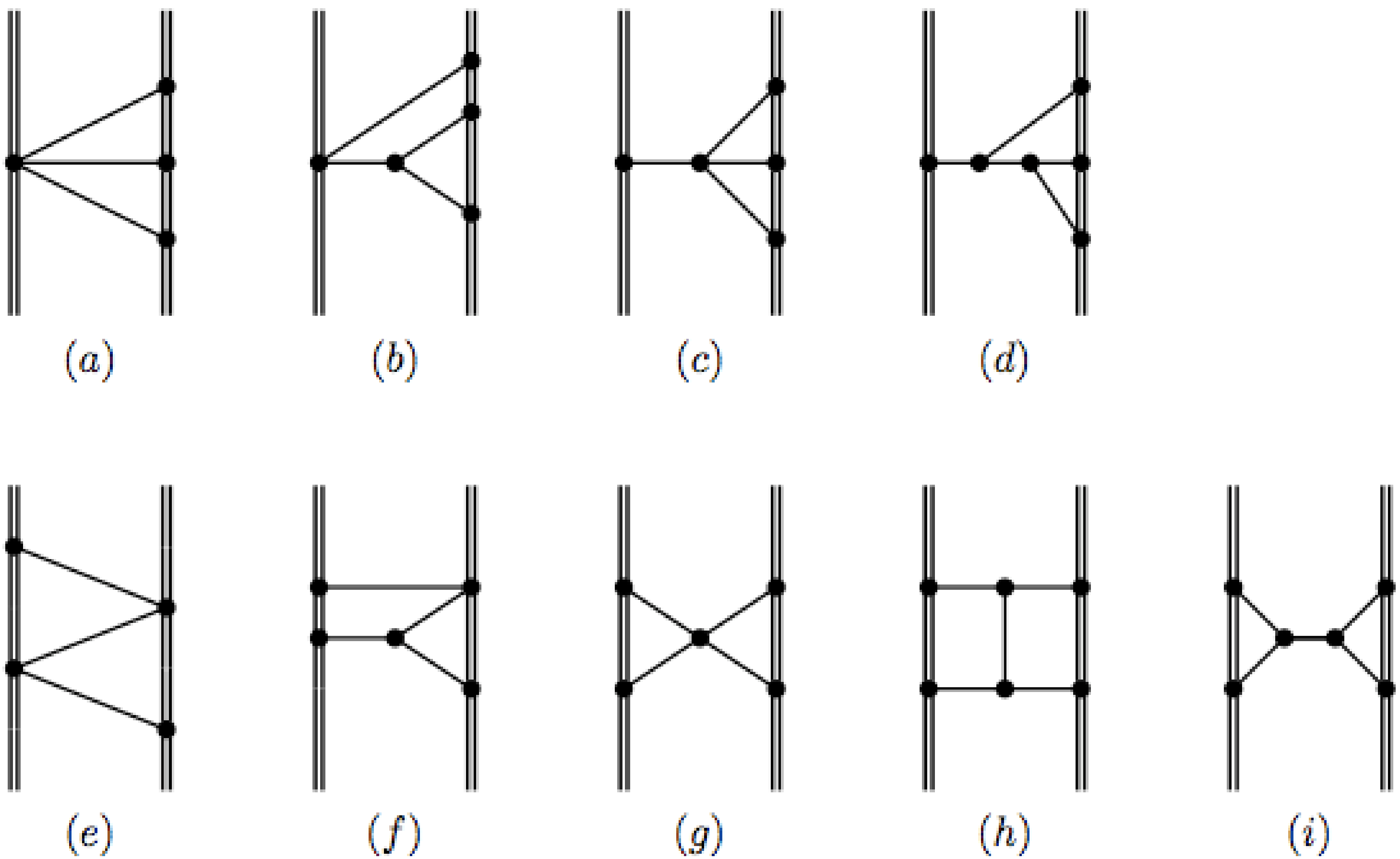
NNLO: $\mathcal{O}(v^4)$ In the EFT (Gilmore + Ross, 2008) this is given by several Feynman diagram topologies (parametrization of $H_{\mu\nu}$ due to Kol + Smolkin, 2007)



$\sim G_N v^4$



$\sim G_N^2 v^2$



$\sim G_N^3 v^0$

Graphs with $G_N^2 v^2$ topology involve finite one-loop Feynman integrals in

1. Euclidean signature, space dimension $d = 3$
2. Massless propagators $\dfrac{1}{\mathbf{k}^2}$ (Newton exchange)

After tensor reduction to scalar integrals, these can be done with the standard formula

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{[(\mathbf{k}+\mathbf{p})^2]^\alpha \, [\mathbf{k}^2]^\beta} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(d/2-\alpha)\Gamma(d/2-\beta)}{\Gamma(d-\alpha-\beta)} (\mathbf{p}^2)^{d/2-\alpha-\beta}$$

Graphs with $G_N^3 v^0$ involve finite two-loop Feynman integrals of the form

$$I(\mathbf{k}) = \int \frac{d^d\mathbf{p}}{(2\pi)^d} \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{\{\mathbf{k}\cdot\mathbf{p}\mathbf{q}^2, \mathbf{k}\cdot\mathbf{q}\mathbf{p}^2, \mathbf{p^2}\mathbf{q}^2, \mathbf{k}^2\mathbf{p}\cdot\mathbf{q}\}}{D(\mathbf{k},\mathbf{p},\mathbf{q})}$$

w/ $$D(\mathbf{k},\mathbf{p},\mathbf{q}) = \mathbf{p}^2\mathbf{q}^2(\mathbf{k}+\mathbf{q})^2(\mathbf{k}+\mathbf{p})^2(\mathbf{k}+\mathbf{q}+\mathbf{p})^2$$

Only the last integral is genuinely two-loop, but can be reduced to the master one-loop integral

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{[(\mathbf{k}+\mathbf{p})^2]^\alpha\,[\mathbf{k}^2]^\beta}$$

by applying the integration by parts technique (see e.g., Smirnov).

Putting everything together

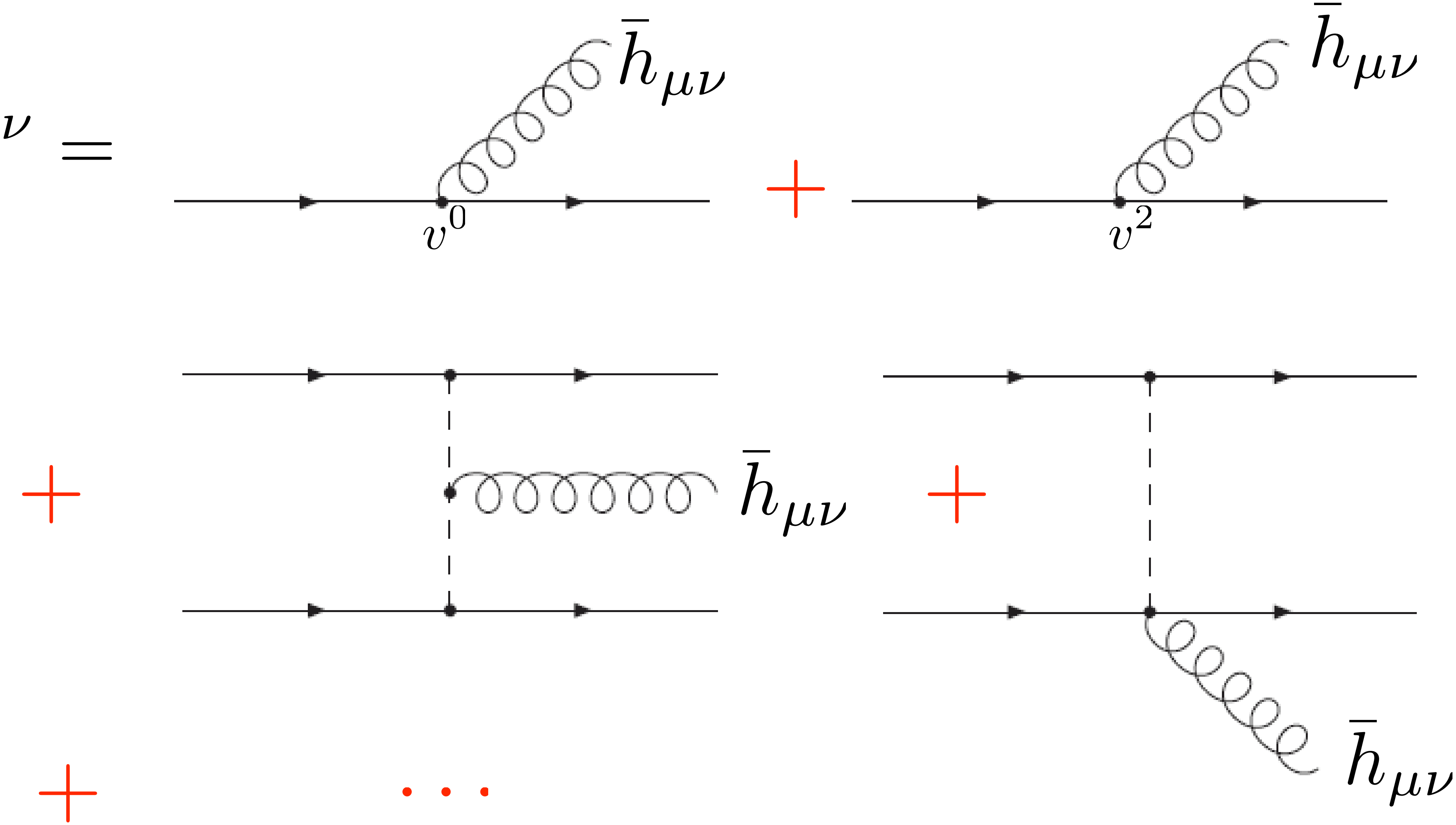
$$
\begin{aligned}
L_{2PN} =& \frac{m_1\mathbf{v}_1^6}{16} \\
&+\frac{Gm_1m_2}{r}\left(\frac{7}{8}\mathbf{v}_1^4-\frac{5}{4}\mathbf{v}_1^2\mathbf{v}_1\cdot\mathbf{v}_2-\frac{3}{4}\mathbf{v}_1^2\mathbf{n}\cdot\mathbf{v}_1\mathbf{n}\cdot\mathbf{v}_2+\frac{3}{16}\mathbf{v}_1^2\mathbf{v}_2^2+\frac{1}{8}(\mathbf{v}_1\cdot\mathbf{v}_2)^2\right. \\
&\qquad\qquad\left.-\frac{1}{8}\mathbf{v}_1^2(\mathbf{n}\cdot\mathbf{v}_2)^2+\frac{3}{4}\mathbf{n}\cdot\mathbf{v}_1\mathbf{n}\cdot\mathbf{v}_2\mathbf{v}_1\cdot\mathbf{v}_2+\frac{3}{16}(\mathbf{n}\cdot\mathbf{v}_1)^2(\mathbf{n}\cdot\mathbf{v}_2)^2\right) \\
&+Gm_1m_2\left(\frac{1}{8}\mathbf{a}_1\cdot\mathbf{n}\mathbf{v}_2^2+\frac{3}{2}\mathbf{a}_1\cdot\mathbf{v}_1\mathbf{n}\cdot\mathbf{v}_2-\frac{7}{4}\mathbf{a}_1\cdot\mathbf{v}_2\mathbf{n}\cdot\mathbf{v}_2-\frac{1}{8}\mathbf{a}_1\cdot\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_2)^2\right) \\
&+Gm_1m_2r\left(\frac{15}{16}\mathbf{a}_1\cdot\mathbf{a}_2-\frac{1}{16}\mathbf{a}_1\cdot\mathbf{n}\mathbf{a}_2\cdot\mathbf{n}\right) \\
&+\frac{G^2m_1m_2^2}{r^2}\left(\frac{7}{4}\mathbf{v}_1^2+2\mathbf{v}_2^2-\frac{7}{2}\mathbf{v}_1\cdot\mathbf{v}_2+\frac{1}{2}(\mathbf{n}\cdot\mathbf{v}_1)^2\right) \\
&+\frac{G^3m_1m_2^3}{2r^3}+\frac{3G^3m_1^2m_2^2}{2r^3}+(1\leftrightarrow 2),
\end{aligned}
$$

NNNLO: Many individual diagrams (~100) but no new Feynman integrals (Gilmore+Ross, in progress)

One graviton sector: Radiation

Must compute the graphs

$T^{\mu\nu} =$



$+ \ \cdots$

(1st graph=LO. Last three graphs are NLO).

This gives the multipole moments of 2-body system

E-quad B-quad E-octo

$$\Gamma[\bar{h}]_{1\partial^2} = \frac{1}{2m_{Pl}} \int dx^0 \left[E^{ij} R_{0i0j} + \frac{4}{3} B^{i,jk} R_{0jik} + \frac{1}{3} E^{ijk} \partial_k R_{0i0j} \right]$$

For example, the quadrupole moment to NLO:

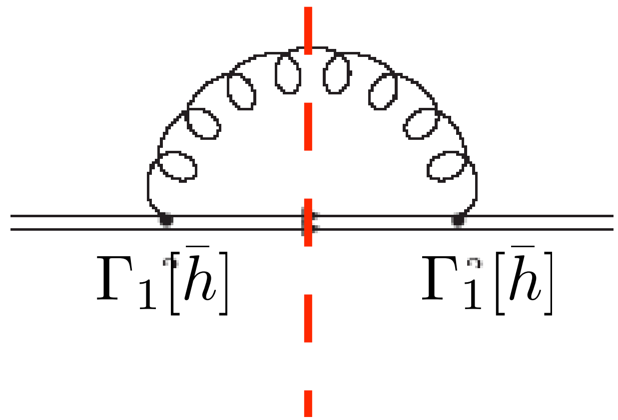
$$E^{ij} = \int d^3\mathbf{x} \left[T^{00} + T^{aa} + \frac{11}{42} \mathbf{x}^2 \ddot{T}^{00} - \frac{4}{3} \dot{T}^{0k} x^k \right] \left[x^i x^j \right]^{TF} + \mathcal{O}(v^4)$$

$$= \sum_a m_a \mathbf{x}_a^i \mathbf{x}_a^j \left[1 + \frac{3}{2} \mathbf{v}_a^2 - \sum_b \frac{G_N m_b}{|\mathbf{x}_a - \mathbf{x}_b|} \right] + \frac{11}{42} \sum_a m_a \frac{d^2}{dt^2} (\mathbf{x_a}^2 \mathbf{x}_a^i \mathbf{x}_a^j)$$

$$- \frac{4}{3} \sum_a m_a \frac{d}{dt} (\mathbf{x_a} \cdot \mathbf{v}_a \mathbf{x}_a^i \mathbf{x}_a^j) - \text{traces} + \mathcal{O}(v^4)$$

Can now use these moments to compute observables, e.g. radiated power to NLO. Use optical theorem to get

Im



$$\frac{dE}{dt} = \frac{G_N}{5}\langle\left(\frac{d^3}{dt^3}E^{ij}(t)\right)^2\rangle + \frac{16G_N}{45}\langle\left(\frac{d^3}{dt^3}B^{ij}(t)\right)^2\rangle + \frac{G_N}{189}\langle\left(\frac{d^4}{dt^4}E^{ijk}(t)\right)^2\rangle + \cdots$$

E.g, binary system in circular orbit

$$\frac{dE}{dt} = \frac{32}{G_N}\left(\frac{\mu}{M}\right)^2 v^{10}\left[1 - \frac{1247}{336}v^2 + \cdots\right]$$

Radiative (long dist.) corrections:

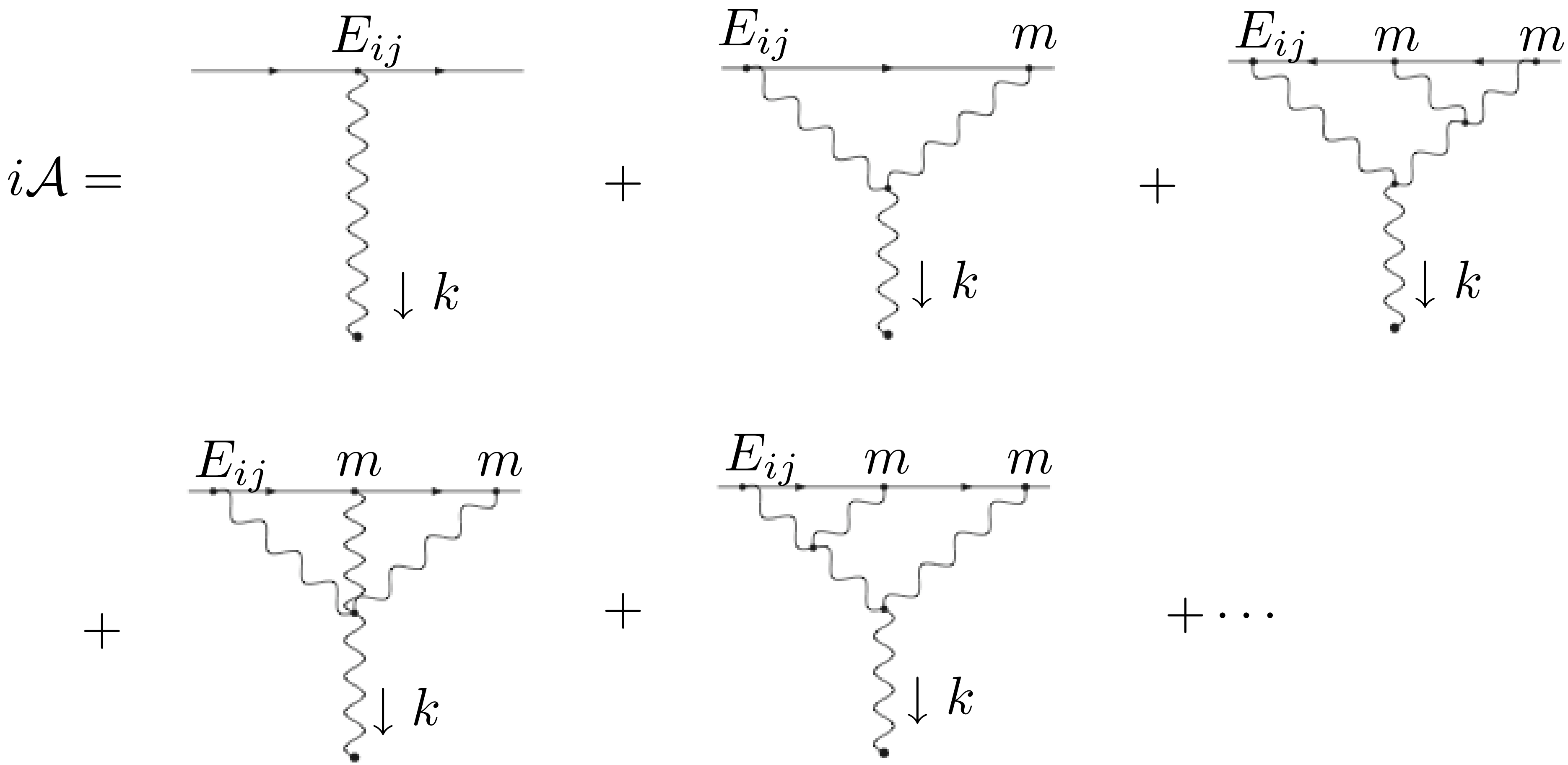
Can use EFT to compute radiative corrections. Everything is calculable in terms of the radiation Lagrangian obtained by matching

$$S[\bar{h}] = S_{EH}[\bar{h}] + S_{GF} + \Gamma[\bar{h}]$$

$$\Gamma[\bar{h}] \supset \frac{1}{2m_{Pl}} \int dx^0 \left[E^{ij} R_{0i0j} + \frac{4}{3} B^{i,jk} R_{0jik} + \frac{1}{3} E^{ijk} \partial_k R_{0i0j} \right] + \cdots$$

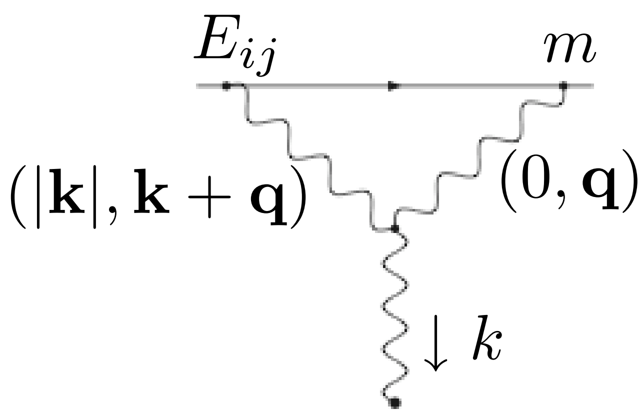
All calcs. can be done in terms of this Lagrangian, for arbitrary moments (not just those obtained by matching to the NR limit $v \ll 1$)

Example: Quadrupolar graviton emission. Amplitude is



Non-linear interaction of emitted gravitons with multipole moments introduces both UV and IR divergences.

Leading IR divergence:



Can be reduced to to scalar integrals of the form

$$I_n(|\mathbf{k}|) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{(\mathbf{q}^2)^n}{\mathbf{k}^2 - (\mathbf{k}+\mathbf{q})^2 + i\epsilon}$$

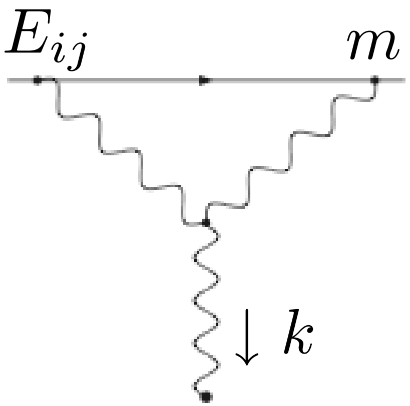
$$(n \geq -1)$$

Note that for $n = -1$ this has an infrared divergence.
As $\mathbf{q} \to 0$ $(d = 3 - 2\epsilon)$

$$I_{-1}(|\mathbf{k}|) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{\mathbf{q}^2} \frac{1}{\mathbf{k}^2 - (\mathbf{k}+\mathbf{q})^2 + i\epsilon} \sim \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{\mathbf{q}^2(\mathbf{k}\cdot\mathbf{q})} \sim \frac{1}{\epsilon_{IR}}$$

Physically, this is the familiar "Coulomb" singularity: nearly on-shell graviton interacts with a long range $1/r$ potential.

The complete result is



$$= i\mathcal{A}_{LO}$$

$$\times[-2iG_N m|\mathbf{k}|]\left(-\frac{\mathbf{k}^2+i\epsilon}{\pi\mu_{IR}^2}e^{\gamma_E}\right)^{-\epsilon}\left[\frac{1}{2\epsilon_{IR}}+\frac{11}{12}+2\epsilon\left(\frac{\pi^2}{16}+\frac{203}{144}\right)\right]$$

$$\left(\mathcal{A}_{LO}=\frac{\mathbf{k}^2}{4m_{Pl}}\epsilon^*_{ij}E_{ij}(|\mathbf{k}|)\right)$$

Note that to order $G_N m|\mathbf{k}|\sim v^3$, the IR singularities drop from $|\mathcal{A}|^2$

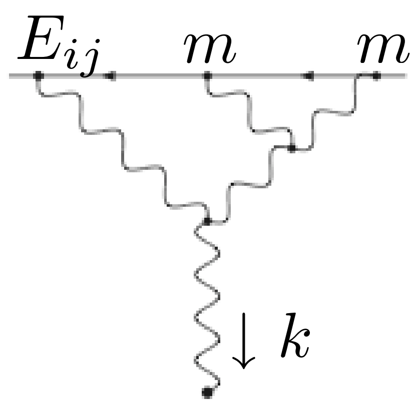
$$\left|\frac{\mathcal{A}}{\mathcal{A}_{LO}}\right|^2=1+2G_N m|\mathbf{k}|+\mathcal{O}(1/\epsilon_{IR}^2)$$

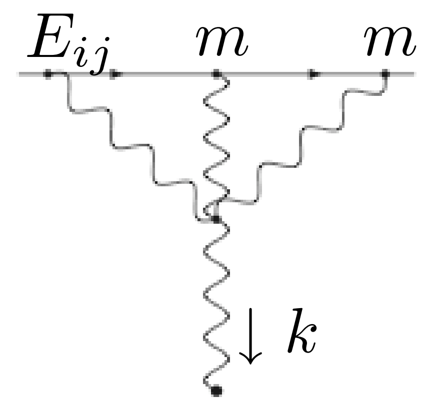
The "Coulomb tail" is responsible for non-analytic corrections to the radiated power in gravitons. Eg,

$$P_{v^3}=P_{LO}\times(4\pi v^3) \qquad P_{v^5}=P_{LO}\times\left(-\frac{8191}{672}\pi v^5\right)$$

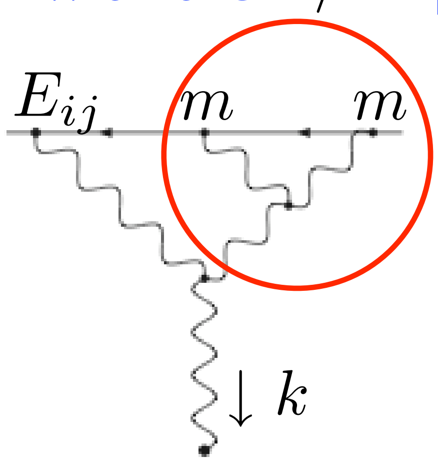
Subleading IR, Leading UV:

The following graphs at NNLO





are UV divergent. This reflects the interaction of nearly on-shell outgoing graviton with the $1/r^2$ potential of the two-body system. Eg.



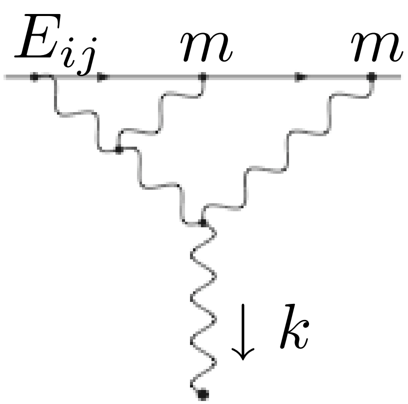
$$\sim \frac{1}{|\mathbf{q}|}$$

$$\int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{|\mathbf{q}|}\frac{1}{\mathbf{k}^2-(\mathbf{k}+\mathbf{q})^2+i\epsilon}$$

$$\sim \int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{|\mathbf{q}|^3} \sim \frac{1}{\epsilon_{UV}}$$

These (two-loop) integrals are fairly straightforward to do, as they correspond to nested one-loop integrals after scalar reduction.

The more challenging integral is



which has both UV and IR divergences. Scalar reduction turns this into the set of integrals

$$
\begin{aligned}
I(n_1, n_2, n_3)(|\mathbf{k}|) \quad &= \quad \int \frac{d^d\mathbf{q}}{(2\pi)^d}\,\frac{d^d\mathbf{p}}{(2\pi)^d}\,\frac{1}{\mathbf{k}^2-(\mathbf{k}+\mathbf{q})^2+i\epsilon}\,\frac{1}{\mathbf{k}^2-(\mathbf{k}+\mathbf{q}+\mathbf{p})^2+i\epsilon} \\
&\times \left(\frac{1}{\mathbf{q}^2}\right)^{n_1}\left(\frac{1}{(\mathbf{k}+\mathbf{q})^2}\right)^{n_2}\left(\frac{1}{\mathbf{p}^2}\right)^{n_3}
\end{aligned}
$$

Basic strategy for "master integral" $I(n_1, n_2, n_3)$:

1. Introduce a single Mellin-Barnes transform

$$\frac{1}{\mathbf{k}^2 - (\mathbf{k}+\mathbf{l})^2 + i\epsilon} = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \Gamma(1+z)\Gamma(-z) \frac{(\mathbf{k}^2 + i\epsilon)^z}{((\mathbf{k}+\mathbf{l})^2)^{z+1}}$$

2. Do "inner" momentum integration in terms of elementary integral

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{[(\mathbf{k}+\mathbf{p})^2]^\alpha \, [\mathbf{k}^2]^\beta}$$

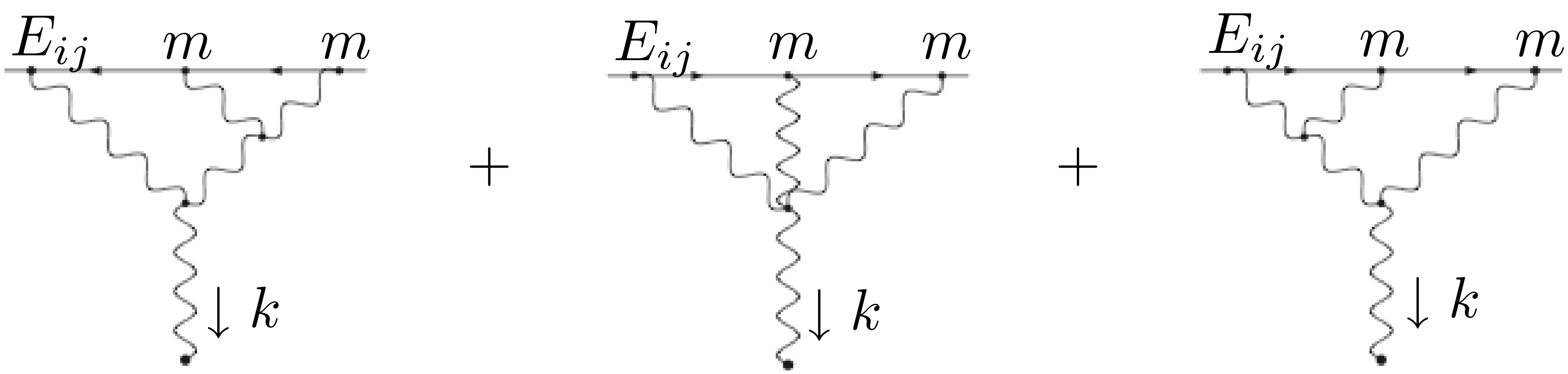
3. Do "outer" momentum integration in terms of elementary integral

$$I_n(|\mathbf{k}|) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{(\mathbf{q}^2)^n}{\mathbf{k}^2 - (\mathbf{k}+\mathbf{q})^2 + i\epsilon}$$

4. Evaluate Mellin-Barnes contour by residues. This gives a product of Gamma function ratios

$$_4F_3(\{n_i, d\}, z=1) \to {}_2F_1(\{n_i, d\}, z=1) \to \prod \Gamma(\{n_i, d\})$$

Result is



$$= i\mathcal{A}_{LO} \times (G_N m|\mathbf{k})^2 \left(-\frac{\mathbf{k}^2 + i\epsilon}{\pi \mu_{IR}^2} e^{\gamma_E} \right)^{-2\epsilon} \left[-\frac{1}{2\epsilon_{IR}^2} - \frac{278}{210}\frac{1}{\epsilon} - \frac{7}{12}\pi^2 - \frac{1777}{14700} \right]$$

and $|\mathcal{A}|^2$ to order $(G_N m|\mathbf{k}|)^2 \sim v^6$

$$\left| \frac{\mathcal{A}}{\mathcal{A}_{LO}} \right|^2 = 1 + 2\pi G_N m|\mathbf{k}| + (G_N m|\mathbf{k}|)^2 \left[\frac{214}{105}\frac{1}{2\epsilon_{UV}} - \frac{214}{105}\left(\gamma_E + \ln \frac{\mathbf{k}^2}{\pi \mu^2} \right) + \frac{4}{3}\pi^2 + \frac{634913}{44100} \right]$$

Infrared divergences cancel to this order!

Interpretation of UV poles:

UV pole in radiative correction to $|\mathcal{A}|^2$ is cancelled by renormalization of multipole moments. Since the subtraction scale μ is arbitrary, moments are scale dependent in the EFT.

$$\mu \frac{d}{d\mu} E_{ij}(\mathbf{k},\mu) = -\frac{214}{105}(G_N m|\mathbf{k}|)^2 E_{ij}(\mathbf{k},\mu)$$

or

$$E_{ij}(\mathbf{k},\mu) = e^{-\frac{214}{105}(G_N m|\mathbf{k}|)^2 \ln\frac{\mu}{\mu_0}} \times E_{ij}(\mathbf{k},\mu_0)$$

To minimize logs in the amplitude, $\mu = |\mathbf{k}|$, while matching scale is $\mu_0 \sim 1/r$
This gives terms $\sim \ln v$ in observables.

Note that in the full theory, the amplitude is finite. Dependence on $1/\epsilon_{UV}$ is cancelled by singularities appearing in the multipole expansion at v^6

(not yet computed)

Conclusions

Gravitational wave processes are a new setting for familiar tools

1. EFTs: Separation of scales.

2. Multi-loop Feynman integral techniques.

3. Summation of IR/UV non-analytic terms.

Here, presented the non-relativistic expansion, but the methods work in other kinematic limits of interest to LIGO/LISA, with a suitably modified power counting scheme (eg, $\lambda = m/M \ll 1$)