Renormalization Of Beta Decay At 3+ Loops

PRESENTED BY PETER VANDER GRIEND

ON BEHALF OF KAUSHIK BORAH, RICHARD J. HILL, RYAN PLESTID

CIPANP | MADISON WISCONSIN | JUNE 2025

ARXIV:2309.15929 ,
ARXIV:2309.07343 ,
ARXIV:2402.13307 ,
ARXIV:2402.14769 .

Caltech

Neutrino Theory Network

PART 1

EFT & β DECAY

- Motivation & relevance for **fundamental physics**.
- Necessary **precision**, and requisite **loop orders**.

PART 2

FERMI FUNC.

- **Point-like** EFT of nuclei and leptons.
- The **Fermi function** from loops.

PART 3

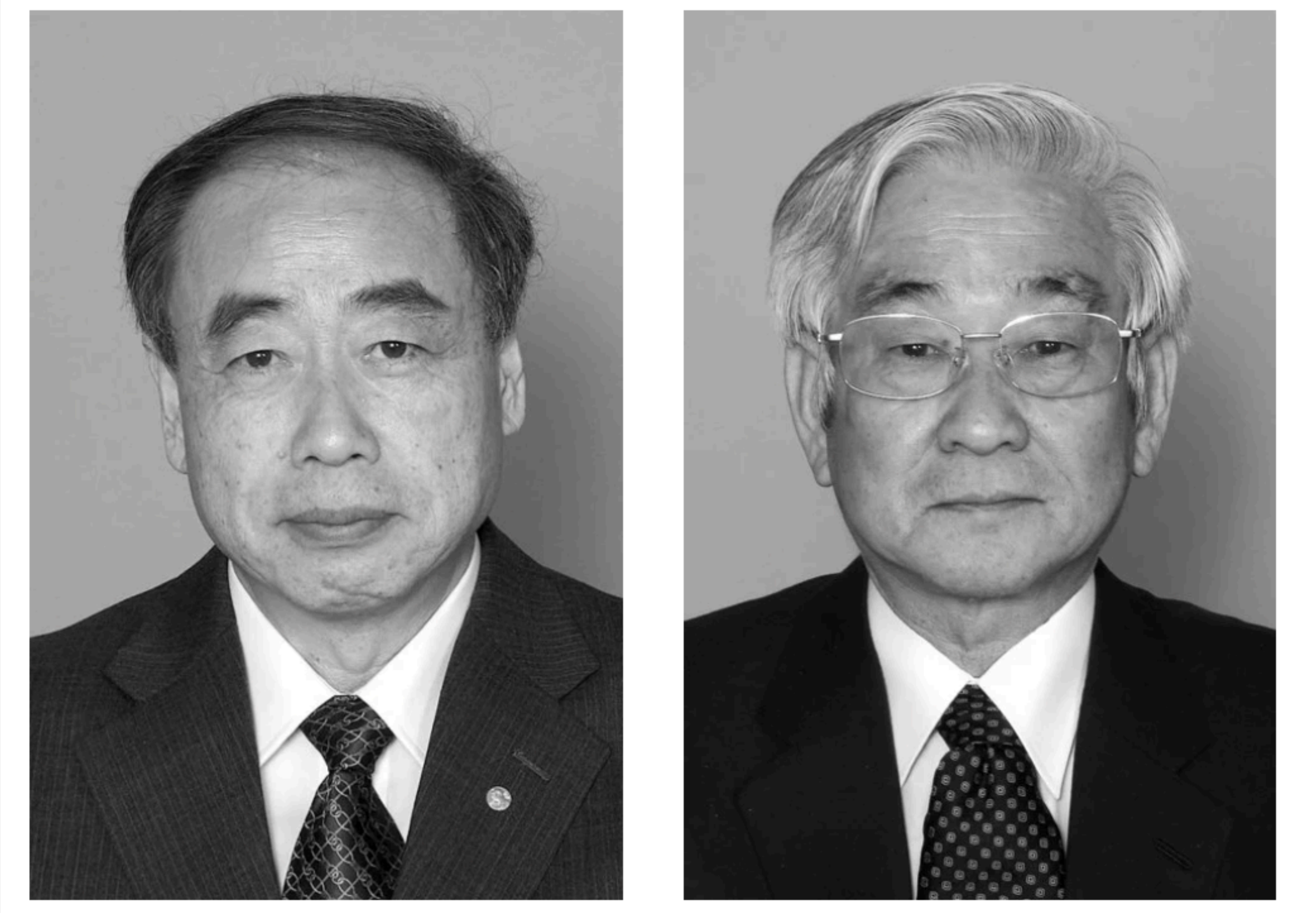
RAD. CORR.

- Structure of **radiative corrections** from EFT.
- Renormalization group **resummation of logarithms**.

Quark Mixing In The SM

FUNDAMENTAL CONSTANTS OF NATURE

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$



CKM ≡ CABIBBO–KOBAYASHI–MASKAWA

Quark Mixing In The SM

BETA DECAY

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

CKM ≡ CABIBBO–KOBAYASHI–MASKAWA

CKM Unitarity

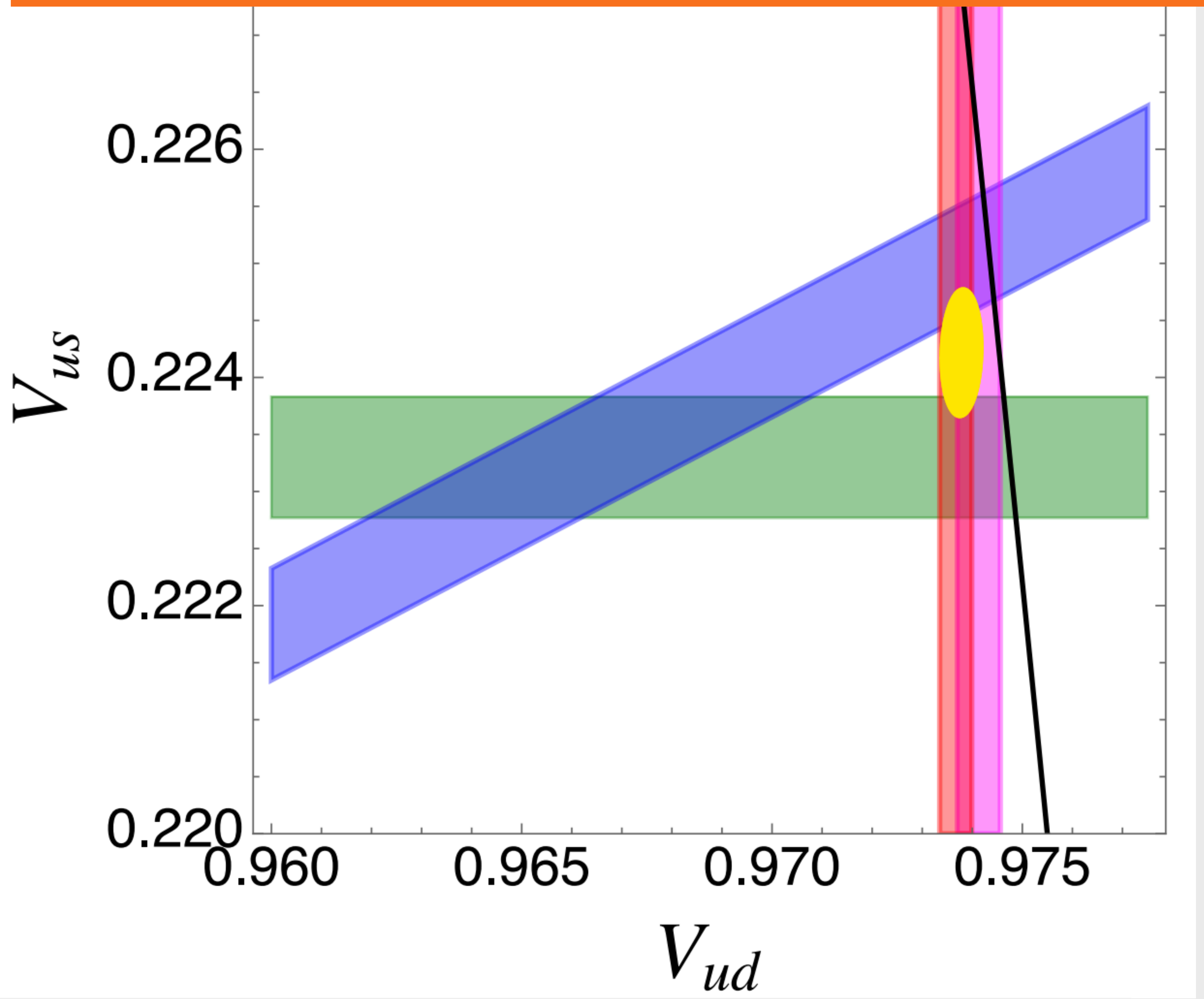
FIRST ROW UNITARITY

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

IN WOLFENSTEIN NOTATION

$$1 - \lambda_{ud}^2 + \lambda_{us}^2 + O(\lambda^6) = 1$$

CIRIGLIANO +++ ARXIV:2208.11707



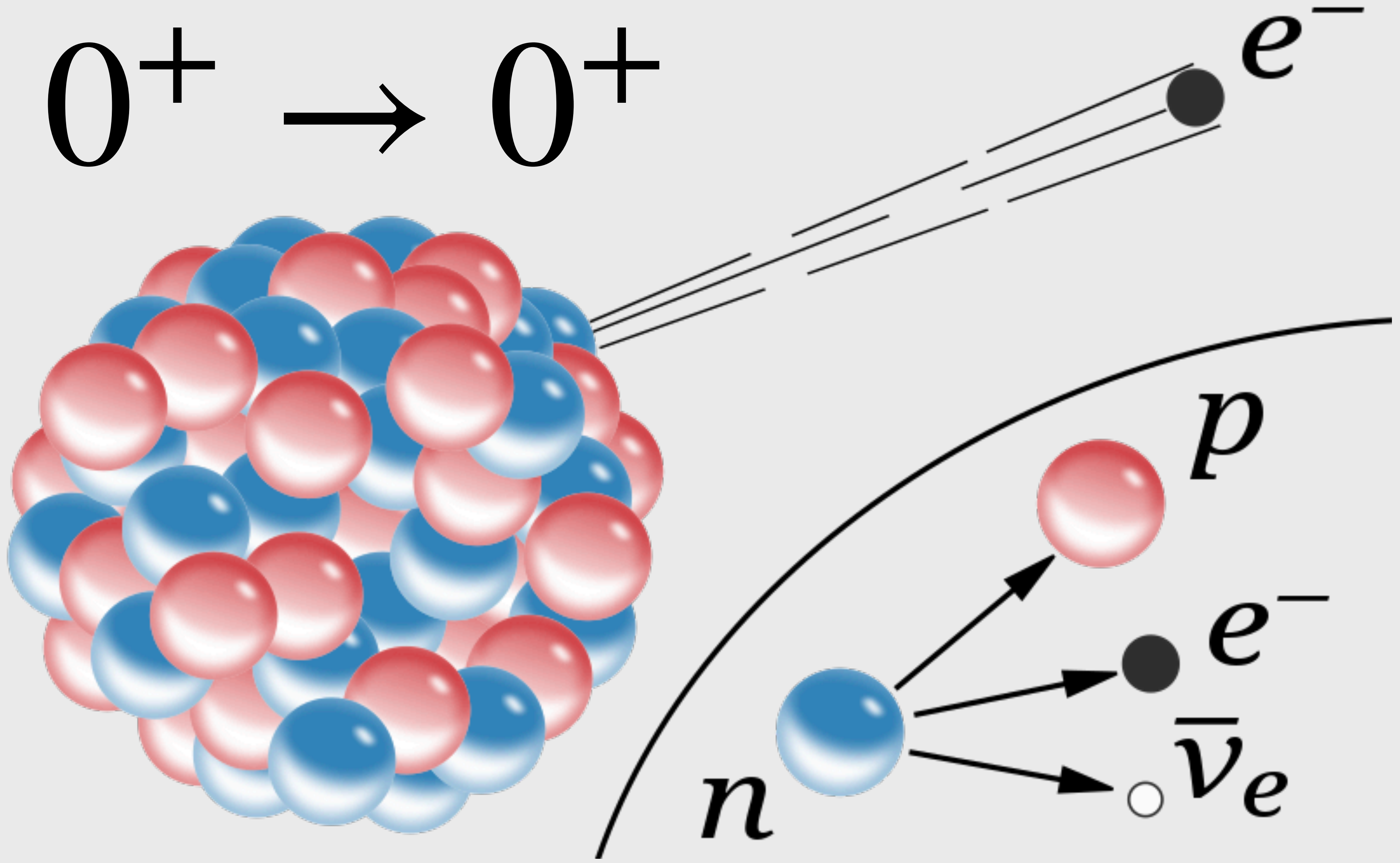
- Tension in first-row CKM unitarity at $\sim 10^{-3}$ level.
- ***If*** theory is under control: strong hints of new physics!

Calculate Matrix Element To High Order

- At this level of accuracy
- Need control over corrections in low-energy theory ***at least*** at

  $O(Z^2\alpha^3)$ i.e. 3+ loops

- **Enhancements from charge of nucleus are important.**



- Precision goal: 100 ppm

Historical Approach

$$\mathcal{F}t \equiv ft(1+\delta'_R)(1+\delta_{\rm NS}-\delta_C) = \frac{K}{2G_V^2\left(1+\Delta_R^V\right)},$$

- The "ft" value includes the Fermi function (Dirac w.f.) .
- Δ_R^V is a short-distance correction.
- $\delta_{\rm NS}$ and δ_C are related to nuclear structure.
- δ'_R computed in the "independent particle model".

THIS TALK: A MODEL-INDEPENDENT EFT APPROACH TO BOTH THE FERMI FUNCTION AND δ'_R

Main Idea

- At $\sim \mathrm{MeV}$ energies nuclei appear point-like.
- Long-distance QED corrections can be compute in an EFT.
- At "leading-power" only sensitive to charge of nucleus.

DEGREES OF FREEDOM

1. Heavy nuclei A and B.
2. Electrons.
3. Photons.

- Just like HQET but nuclei are the "heavy quarks".
- Gauge field is photon.

Main Idea

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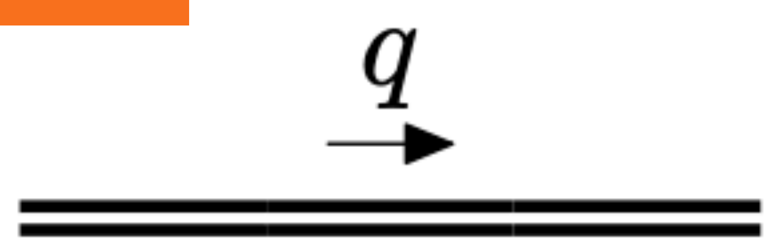
SIMPLE FEYNMAN RULES

COULOMB-FIELD

$$= iZe\,\delta^\mu_0\, 2\pi\delta(q^0)$$

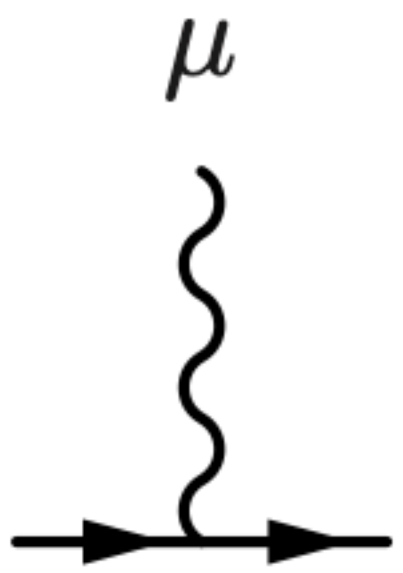
NUCLEI

$$= ie\,\delta^\mu_0$$



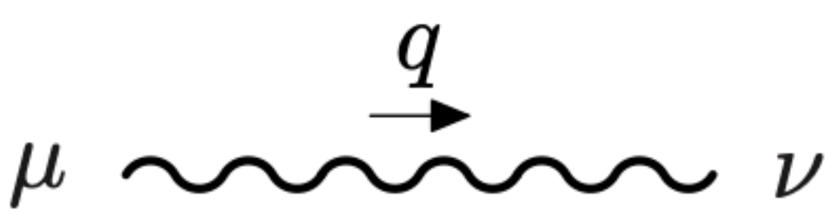
$$= \frac{i}{q^0 + i0}$$

ELECTRONS



$$= -ie\,\gamma^\mu$$

PHOTONS



$$= \frac{-i}{q^2 - \lambda^2 + i0}$$

Impact For Flavour Physics

SHIFTING δ_3

transition	$(\Delta a) \times Z^2 \alpha^3 \log(\Lambda/m)$
$^{14}O \to {}^{14}N$	-1.1×10^{-4}
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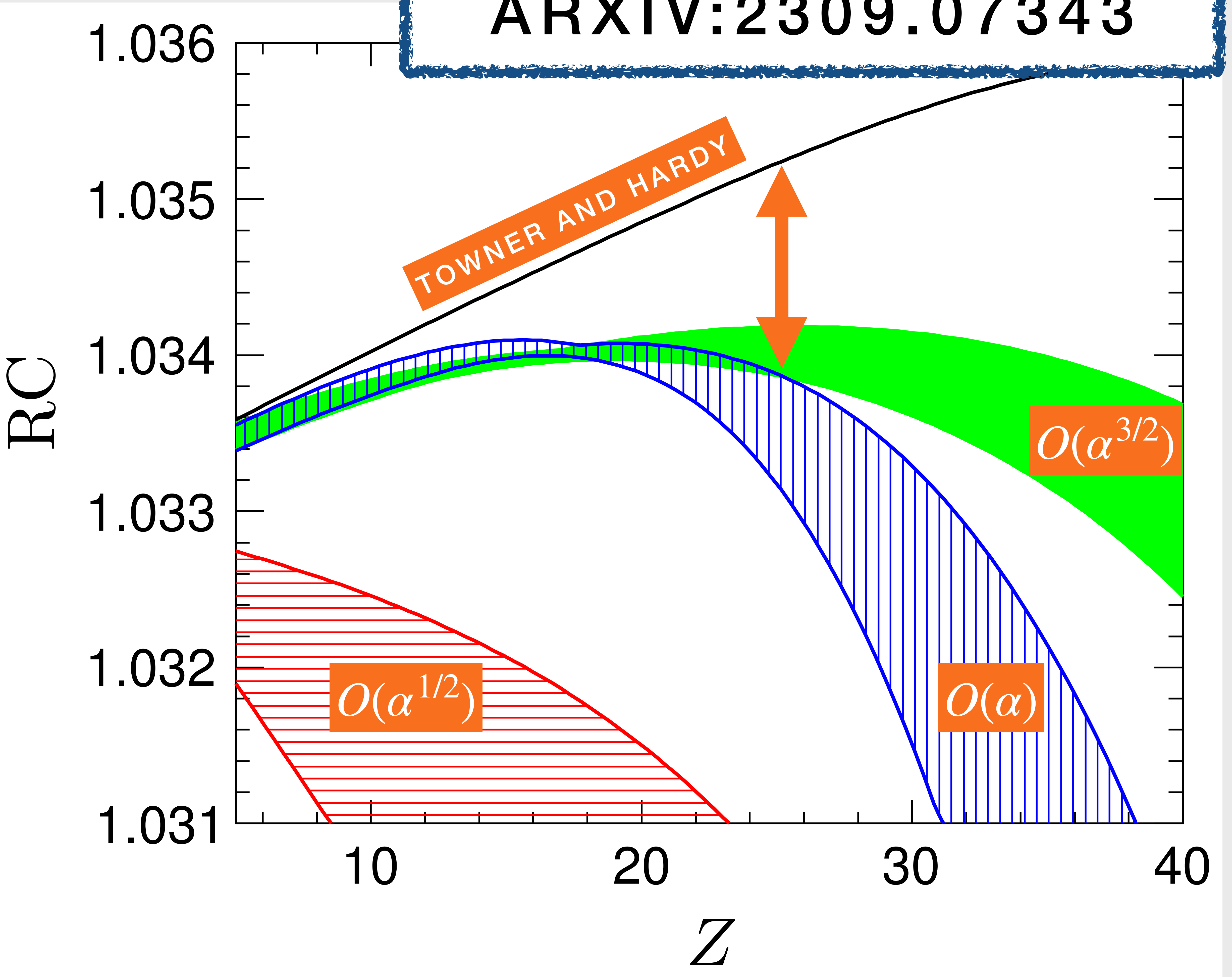

COUNTING $Z \sim \log \sim 1/\sqrt{\alpha}$

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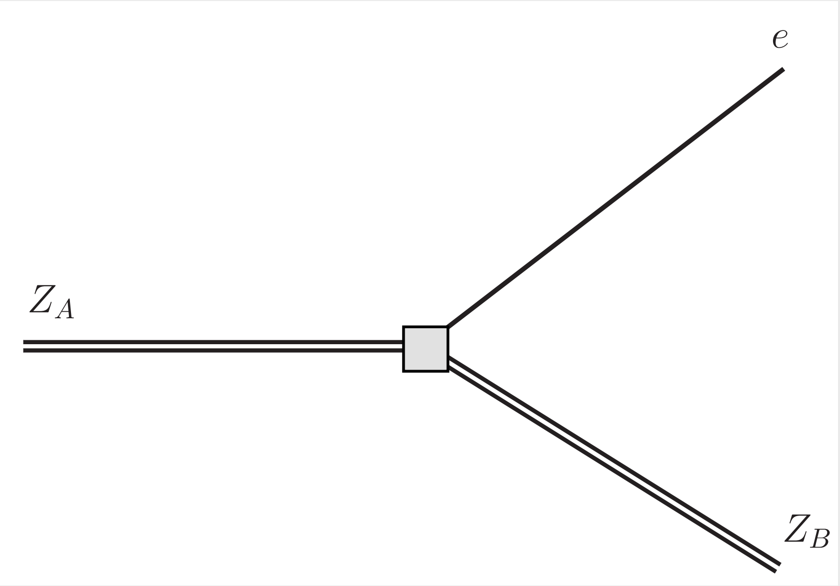
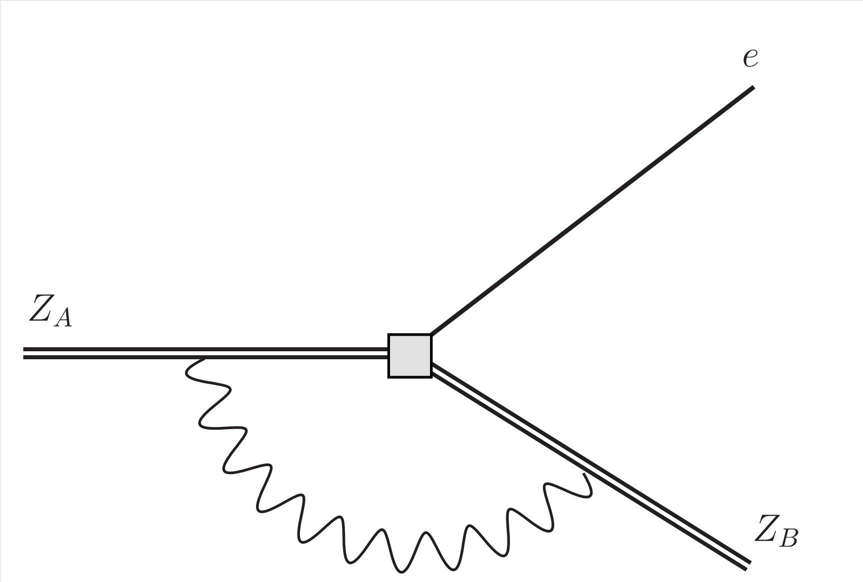
We Just Compute Diagrams!

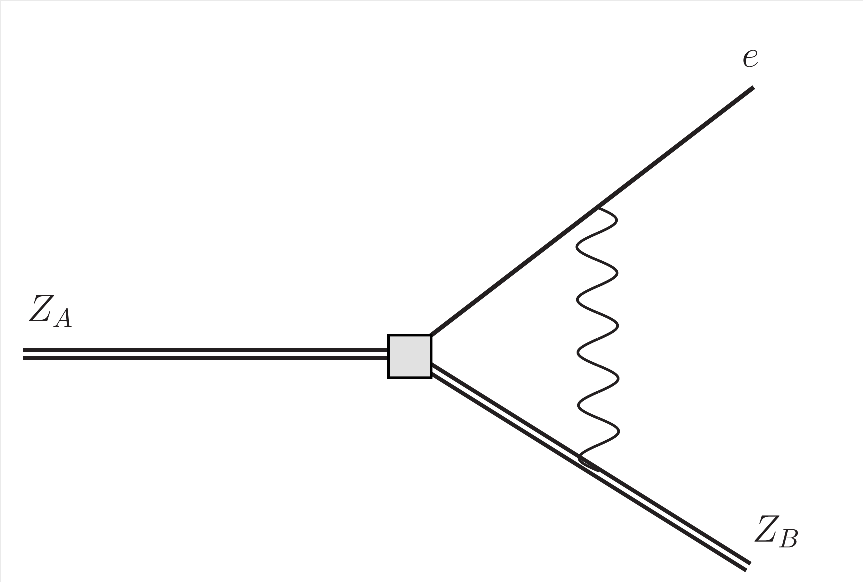
WAVEFUNCTION RENORMALIZATION NOT SHOWN

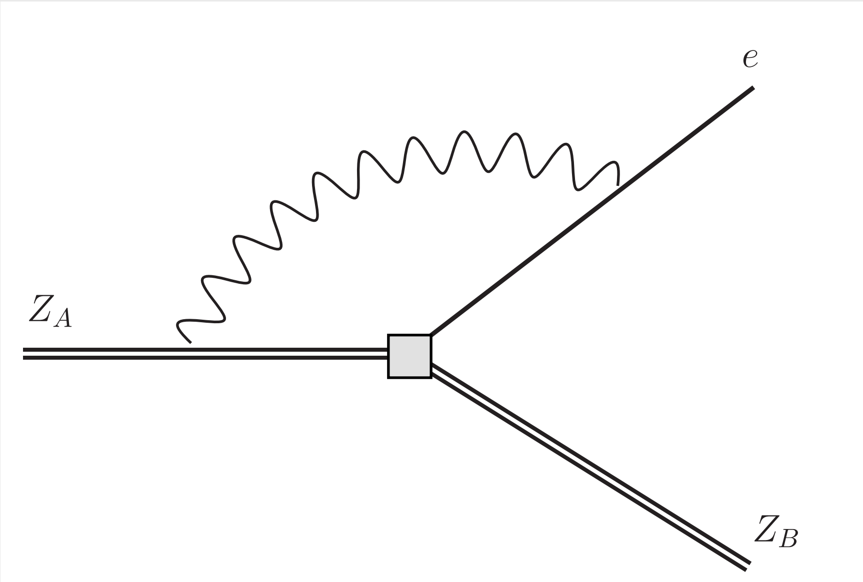
TREE-LEVEL



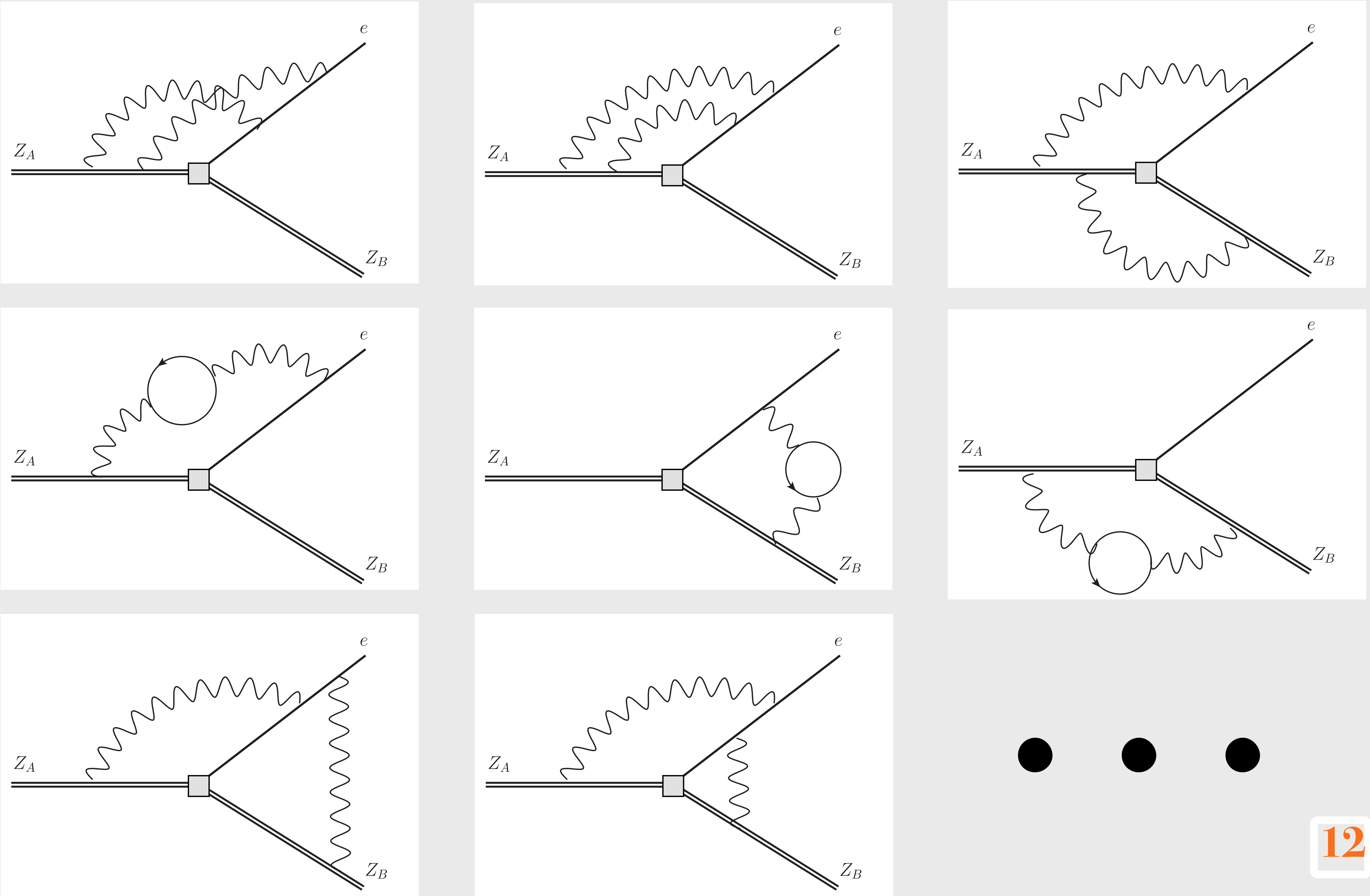
ONE LOOP







TWO LOOP



Number Of Diagrams Grows Factorially

TREE-LEVEL

- 1 diagram.

ONE LOOP

- 3 diagrams.

TWO LOOP

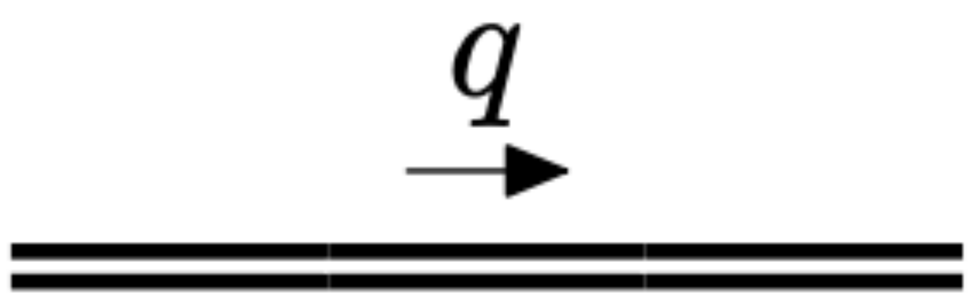
- 21 diagrams.

THREE LOOP

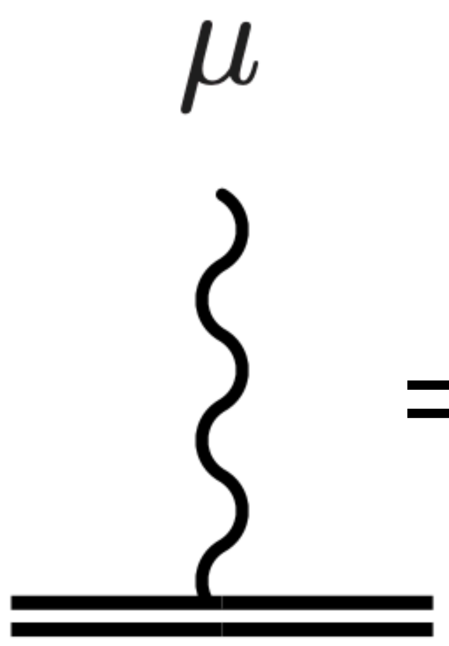
- 144 diagrams.

- For the Fermi function we need 4+ loops.
- This is not feasible by brute force.

Solution : Make Use Of Simplified Feynman Rules



$$= \frac{i}{q^0 + i0}$$



$$= i(Z_A e)\delta^\mu_0$$

Number Of Diagrams Grows Factorially

TREE-LEVEL

- 1 diagram.

ONE LOOP

- 3 diagrams.

TWO LOOP

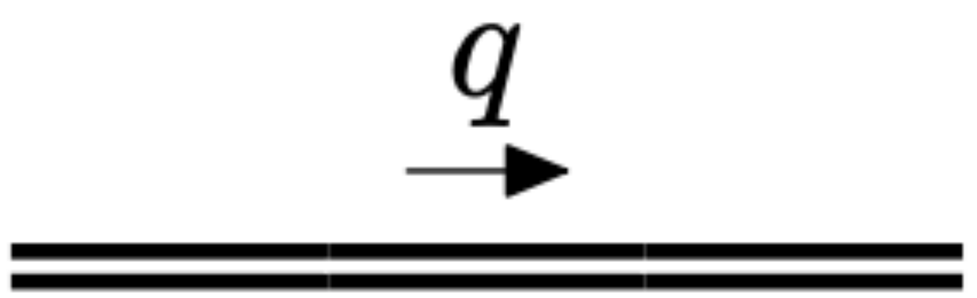
- 21 diagrams.

THREE LOOP

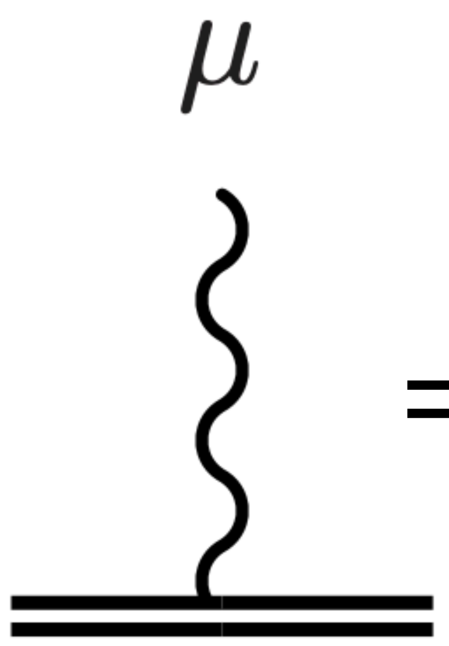
- 144 diagrams.

Reduce Number Of Diagrams
Avoid Difficult Integrals

Solution: Make Use Of Simplified Feynman Rules



$$= \frac{i}{q^0 + i0}$$



$$= i(Z_A e)\delta_0^\mu$$

Equivalent Feynman Rules

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TREE-LEVEL

- 1 diagram.

ONE LOOP

- 2 diagrams.

TWO LOOP

- 5 diagrams.

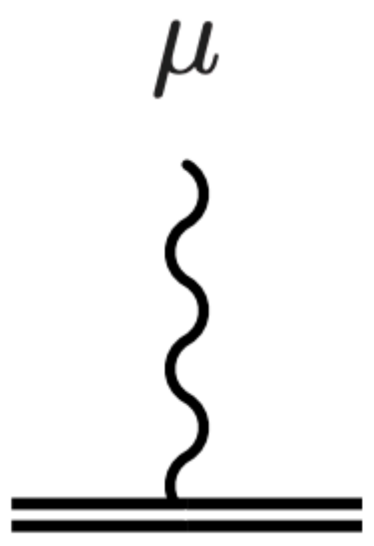
THREE LOOP

- 10 diagrams.

1 NUCLEUS WITH UNIT CHARGE

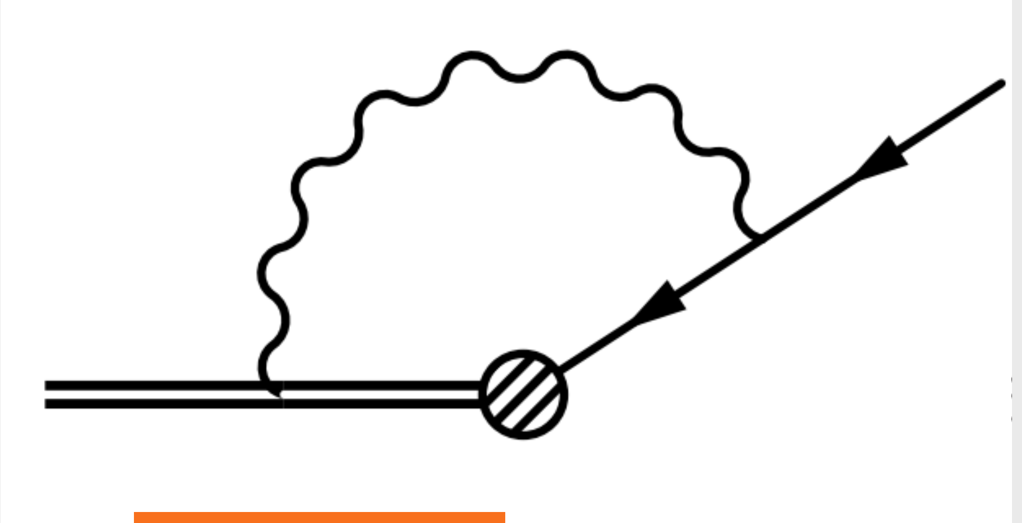
+ A BACKGROUND COULOMB FIELD

$$= iZe\,\delta^{\mu}_{0}\,2\pi\delta(q^0)$$

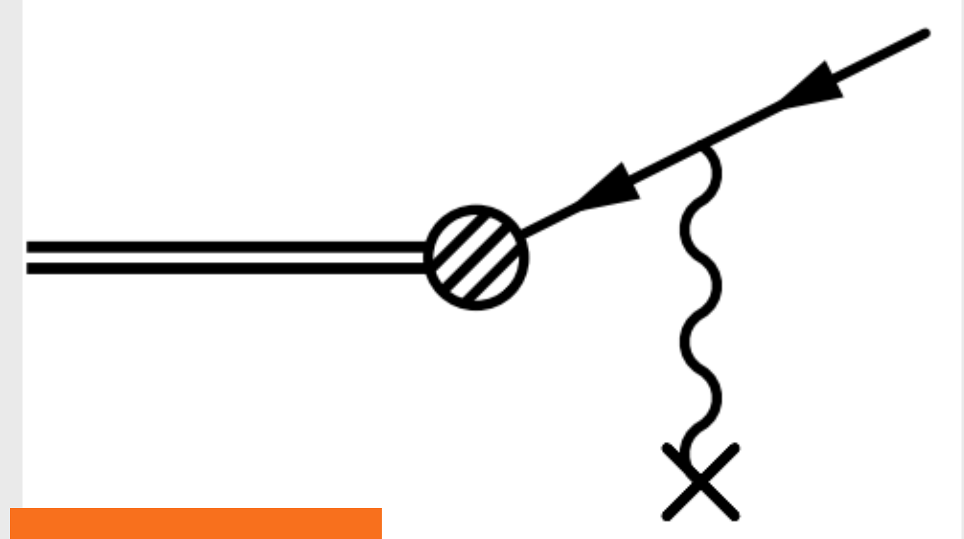


$$= ie\,\delta^{\mu}_{0}$$

ONE LOOP



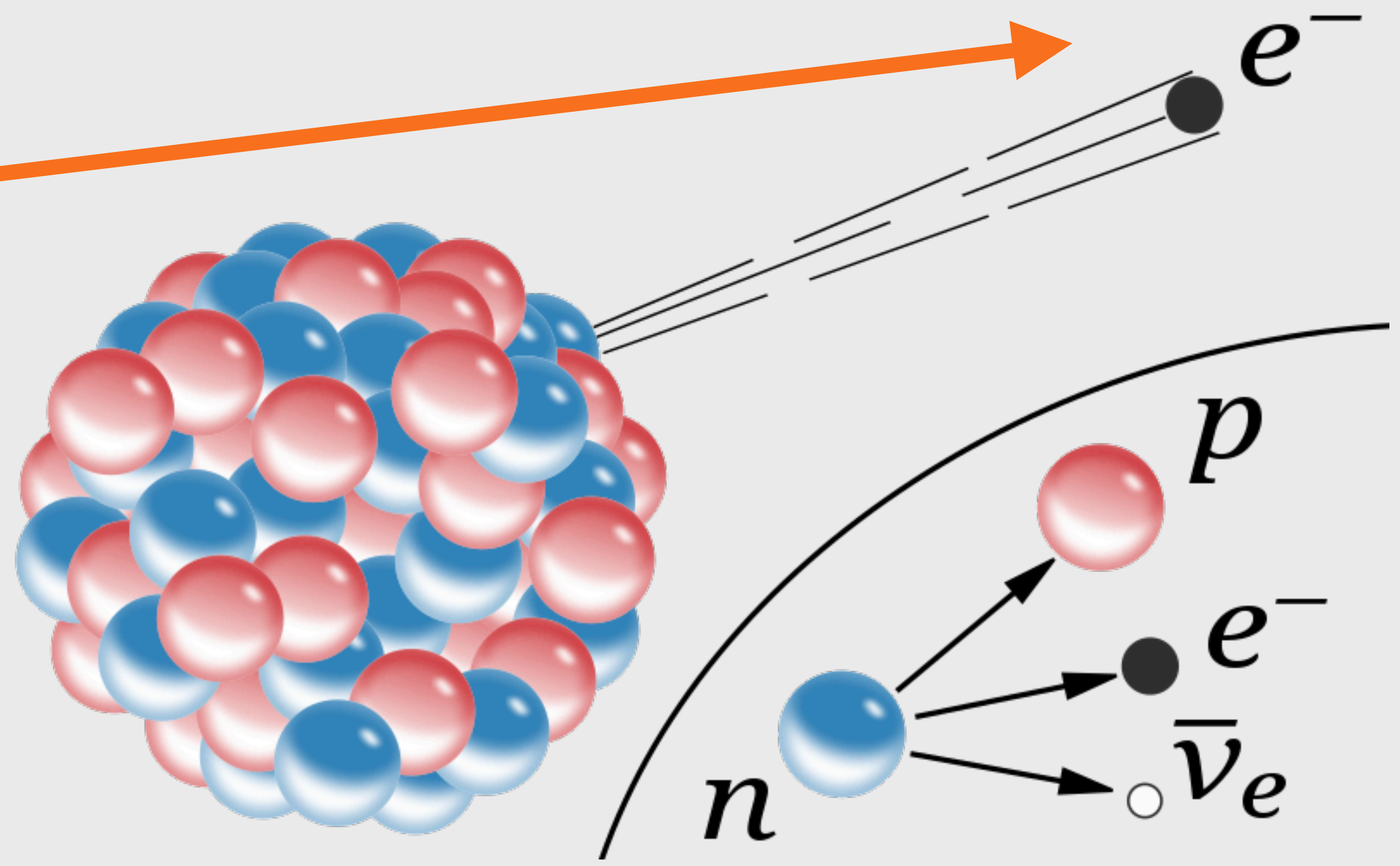
$\mathcal{O}(\alpha)$



$\mathcal{O}(Z\alpha)$

Fermi Function

ATTRACTED TO NUCLEUS



- Coulomb effects are UV divergent.
- Largest effects are a series in $Z\alpha$
- Historically done with finite-distance regulator

$$\langle e^- | \bar{\psi}(\mathbf{x}) | 0 \rangle \sim \left(\frac{1}{|\mathbf{x}|} \right)^{\nu}$$

$$\nu = \sqrt{1 - Z^2\alpha^2} - 1$$

Extraction Of Hard Matrix Element

$$\Psi(\mathbf{x}) = \mathcal{M}_S(\mu_S)\mathcal{M}_H(\mu_S, \mu_H)\tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

KNOWN TO ALL ORDERS IN $Z\alpha$

NEW CALCULATION

EXTRACT WHAT WE WANT

$$\mathcal{M}_H(\mu_S, \mu_H) = \frac{\Psi(\mathbf{x})}{\tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})\mathcal{M}_S(\mu_S)}$$

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Fermi-Function In The $\overline{\text{MS}}$ Scheme

$$\mathcal{M}_H(\mu_S,\mu_H) = \mathrm{e}^{\frac{\pi\xi}{2}+i\xi\left(\log\frac{2p}{\mu_S}-\gamma_\mathrm{E}\right)-i(\eta-1)\frac{\pi}{2}}\frac{2\Gamma(\eta-i\xi)}{\Gamma(2\eta+1)}$$

$$\sqrt{\frac{\eta-i\xi}{1-i\xi\frac{m}{E}}}\sqrt{\frac{E+\eta m}{E+m}}\sqrt{\frac{2\eta}{1+\eta}}\left(\frac{2p\mathrm{e}^{-\gamma_\mathrm{E}}}{\mu_H}\right)^{\eta-1}\times\left[\frac{1+M^*}{2}+\frac{1-M^*}{2}\gamma^0\right]$$

- $\eta = \sqrt{1-Z^2\alpha^2}$
- $\xi = Z\alpha/\beta$
- $M = (E+m)(1+\mathrm{i}\xi m/E)/(E+\eta m)$

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Factorization Theorem

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- Amplitude depends on Wilson coefficient and matrix element.

$$\mathrm{d}\Gamma \propto |C(\mu)|^2 |\mathscr{M}(\mu)|^2 + \mathcal{O}\left((pR)^2\right)$$

- Implies that all ***short-distances*** factorize from ***long-distances***.

$$\mathcal{F}t \equiv ft(1+\delta_R')(1+\delta_{\mathrm{NS}}-\delta_C) = \frac{K}{2G_V^2(1+\Delta_R^V)}$$

EFT Definition Of `Outer' Corrections

$$\tilde{F}(Z,E) = \left[|\mathscr{M}(\mu)|^2 \right]_{\text{leading}-Z\alpha}$$

$$(1+\tilde{\delta}_R) \equiv \frac{\langle |\mathscr{M}(\mu)|^2 \rangle}{\langle \tilde{F}(Z,E) \rangle}$$

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$(1+\delta_{\text{NS}}-\delta_C)(1+\Delta_R^V)$ ABSORBED IN WILSON COEFFICENT

EFT Definition Of `Outer' Corrections

$$(1+\delta'_R) := \left[\frac{C(\mu_L)/C(\mu_H)}{\exp\left[\left(1-\sqrt{1-Z^2\alpha^2}\right)\ \log(\mu_H/\mu_L)\right]}\right]^2 \left(\frac{\int d\Pi \qquad \langle|\mathcal{M}_H|^2\rangle}{\int d\Pi\ F(Z,E)\times\frac{4\eta}{(1+\eta)^2}}\right)_{\mu=\mu_L}$$

$$(1+\tilde{\delta}_R) \equiv \frac{\langle|\mathscr{M}|^2(\mu)\rangle}{\langle\tilde{F}(Z,E)\rangle}$$

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EFT Definition Of Long-Distance Corrections

Resummation With RG + EFT

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$$d\Gamma \propto |C(\mu)|^2 |\mathcal{M}(\mu)|^2$$

No Large Logs

$$= C(\mu_H) \left[\frac{|C(\mu_L)|^2}{|C(\mu_H)|^2} \right] |\mathcal{M}(\mu_L)|^2$$

Calculate With Renormalization Group

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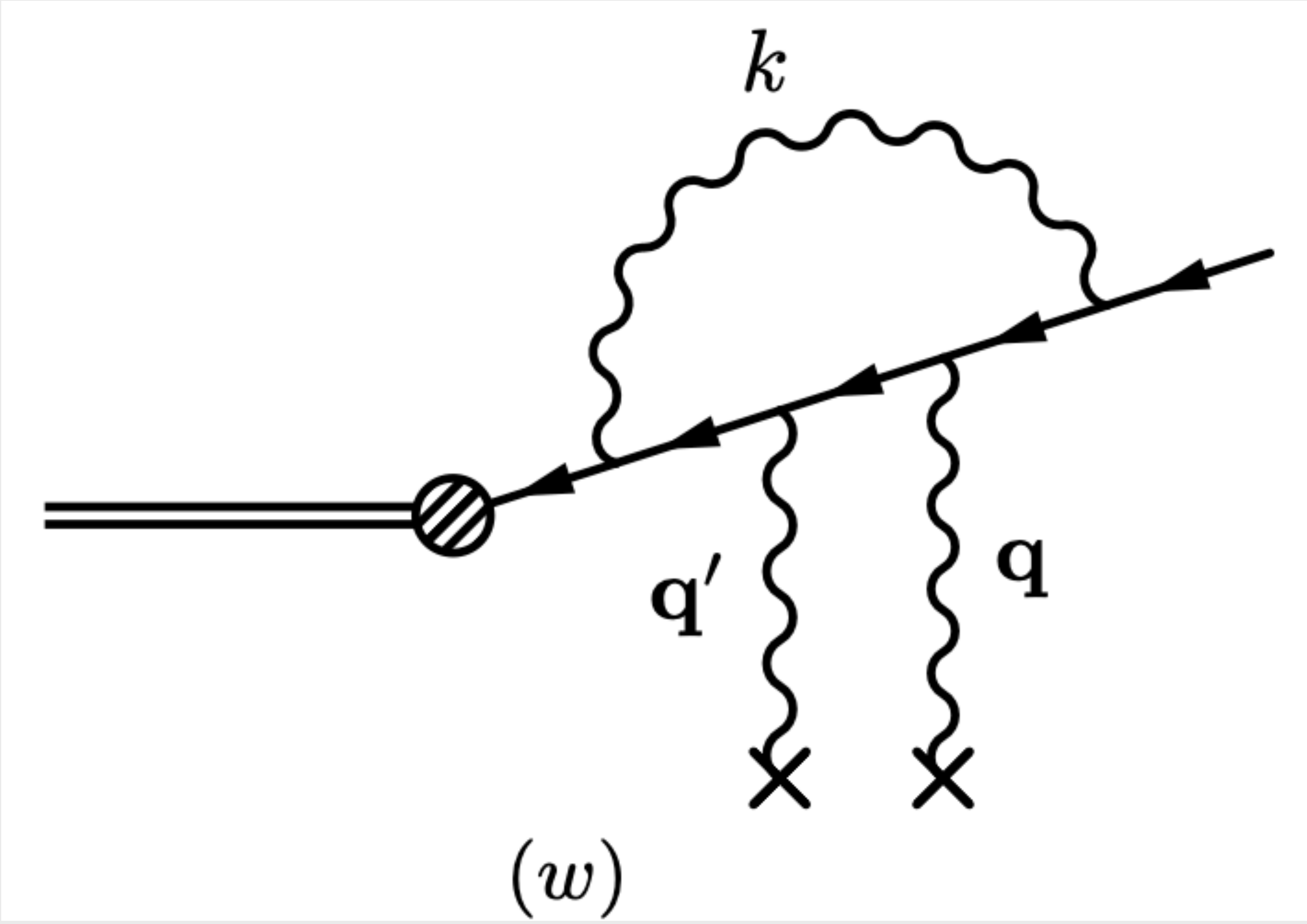
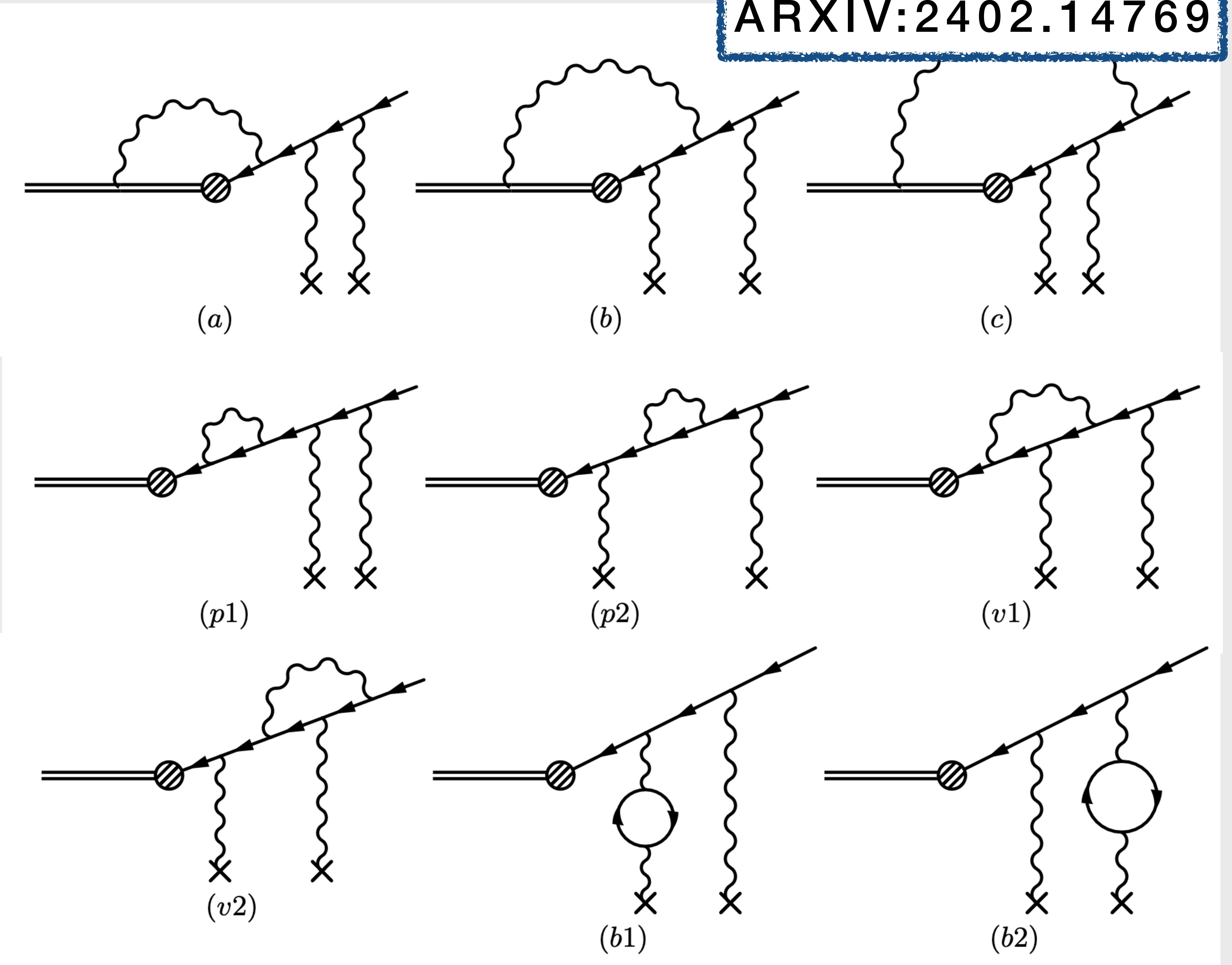
USE EIKONAL ALGEBRA TO REDUCE DIAGRAMS

$$\gamma_2^{(1)} = 16\pi^2\left(6 - \frac{\pi^2}{3}\right)$$





MIXED EUCLIDEAN + LORENTZIAN INTEGRALS

New Result For Anomalous Dimension

Z^n \ Loops	1-loop	2-loop	3-loop	4-loop
Z^0	$\gamma_0^{(1)} = -3$	$\gamma_1^{(2)} = -16\zeta_2 + \frac{5}{2} + \frac{10}{3}n_e$	$\gamma_2^{(3)} =$ GROZIN 2003	$\gamma_3^{(4)} =$ GROZIN 2023
Z^1	$\gamma_0^{(0)} = 0$	$\gamma_1^{(1)} = \gamma_2^{(2)}$	$\gamma_2^{(2)} = \gamma_2^{(1)}$	$\gamma_3^{(3)} = \gamma_3^{(2)} - \gamma_3^{(0)}$
Z^2	–	$\gamma_1^{(0)} = -8\pi^2$	$\gamma_2^{(1)} = 16\pi^2\left(6 - \frac{\pi^2}{3}\right)$	$\gamma_3^{(2)} =$?
Z^3	–	–	$\gamma_2^{(0)} = 0$	$\gamma_3^{(1)} = 2\gamma_3^{(0)}$
Z^4			–	$\gamma_3^{(0)} = -32\pi^4$

RESUMMATION COMPLETE THROUGH 3-LOOPS!

NEW INPUT

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Impact For Flavour Physics

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SHIFTING δ_3

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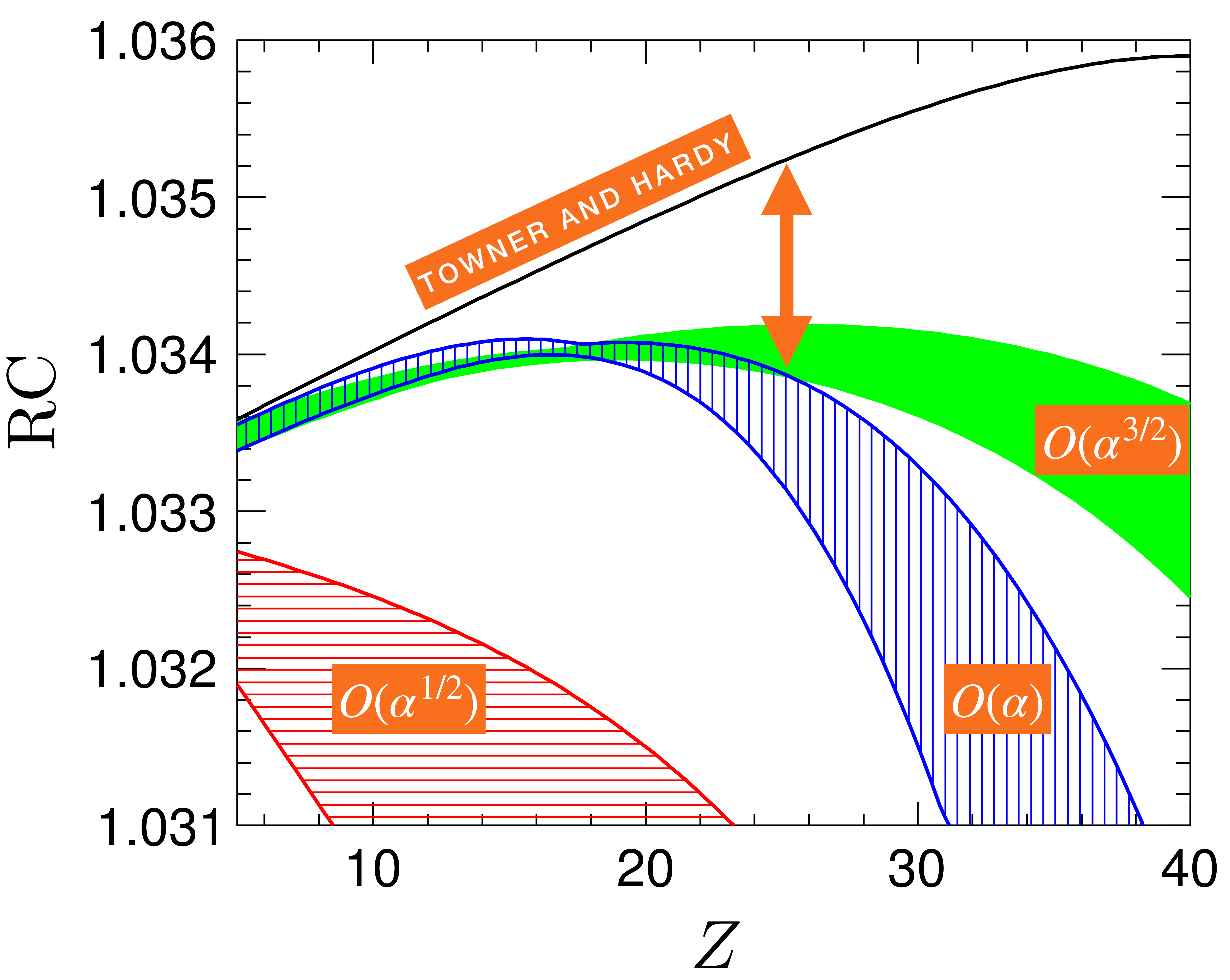

COUNTING $Z \sim \log \quad \sim 1/\sqrt{\alpha}$

What's Next?

Order $Z\alpha^2$ Correction

- We will soon present two-loop finite corrections in the long-distance EFT.
- Supplies new model-independent correction relevant for interpreting superallowed beta decays.

Two-loop corrections for superallowed beta decays

Peter Vander Griend[1,2], Zehua Cao[1], Richard J. Hill[1,2], and Ryan Plestid[3]

TO APPEAR LATER THIS YEAR

Power Corrections

- Theory so far worked out at leading power.
- Sub-leading power is straightforward, but matrix elements must be computed.
- Structure of factorization formulae will change.
- Many simple things to do here.

Matching Calculations

- Chiral-EFT and/or *ab initio* provides a microphysical framework to extract Wilson coefficient.
- Matching calculation still needs to be performed.

Summary

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- Long-distance QED corrections to beta decay are enhanced by nuclear charge.
- Compute in a point-like EFT with short-distance corrections absorbed into low-energy constants.
- First updates to outer radiative corrections in 40 years.
- Impacts CKM unitarity.