Quantum Closures and Riemann Solvers for Neutrino Moment Transport

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How do we undertake supernova and neutron star merger simulations which account for neutrino flavor transformation?

- The issue is oscillations introduce a new length scale
  - The neutrino oscillation wavelength in dense matter is ~10 microns.
  - The best simulations have spatial grid zones ~100 m – 1 km.
  - To make it even worse, the time steps are set by the light crossing time of a grid zone.
- Throwing silicon at the problem is not the solution. To make simulations with transformations feasible we will have to make approximation and/or get creative:
  - e.g. Nagakura & Zaizen rescaled the neutrino Hamiltonian down by a factor of 10^{-4} then extrapolated their results back to the proper strength.

Nagakura & Zaizen PRL **129** 261101 (2022)

see also Xiong et al PRD **107** 083016 (2023)

Oscillations with moments

- Many classical simulation codes, e.g. FLASH, evolve the neutrino field using moments.
  - the number of moments evolved is usually small, e.g. M1 transport evolves 1+3=4 moments in 3D.
- It is possible to compute neutrino transformation with moments.

Strack and Burrows, PRD **71** 093004 (2005)

Zhang and Burrows, PRD **88** 105009 (2013)

Myers et al, PRD **105** 123036 (2022)

Grohs et al, PLB, **846**, 138210 (2023)

Froustey et al, PRD **109** 043046 (2024)

Grohs et al., ApJ **963** 11 (2024)

Froustey et al., **111** 063022 (2025)

Kneller et al., PRD **111** 063046 (2025)

Grohs et al, PRD **111** 083018 (2025)

Froustey arXiv:2505.16961 (2025)


NEW
FERRERO ROCHER
the golden experience
MOMENTS
Make The Moment Perfect

- We define a quantum moment of the single-particle distribution matrix *f* as

$$\mathcal{M}_n(q) = \int q \cos^n\theta \;\; f \, d\Omega_q$$

- Where n is an integer, q is the energy of the neutrino, θ the angle relative to a given direction (e.g. the radial)

- The first few moments have well-known names
  - n = 0 is the energy density E
  - n = 1 is the energy flux F
  - n = 2 is the pressure P

- Assuming spherical symmetry, the moments evolve according to

$$\frac{\partial E}{\partial t}+\frac{\partial F}{\partial r}+\frac{2F}{r}=-i[H_V+H_M+H_E,E]+i[H_F,F]$$

$$\frac{\partial F}{\partial t}+\frac{\partial P}{\partial r}+\frac{3P-E}{r}=-i[H_V+H_M+H_E,F]+i[H_F,P]$$

$$\vdots$$

  - the H's are contributions to the neutrino Hamiltonian,
  - the absorption / emission / collisions have been omitted
- The infinite tower of equations can be truncated at what ever level one desires.
  - Usually one considers two schemes: a one-moment (M0) and a two-moment (M1)
- We need a relationship between the moments to close the equations.
- This relationship is called 'The Closure'

Quantum Closures

- Since we are using quantum moments, we need a Quantum Closure.
- Start from the ansatz that two quantum moments, e.g. E and P, are related by

$$P = L\,E\,R$$

- L and R are arbitrary matrices
- Since E and P are Hermitian they have eigenvalue matrices

$$E = U_E \Lambda_E U_E^{\dagger} \qquad P = U_P \Lambda_P U_P^{\dagger}$$

- the U's are unitary 'mixing' matrices, the Λ's are the matrices of the eigenvalues.

- We can factorize U_E and U_P as

$$U_E = \Upsilon_E \Phi_E \qquad\qquad U_P = \Upsilon_P \Phi_P$$

- where Υ_E and Υ_P are unique unitary matrices which can be written in terms of the elements of E and P, and Φ_E and Φ_P are arbitrary diagonal unitary matrices.

- Define the quantum Eddington factor matrix to be

$$X = \Lambda_P \Lambda_E^{-1}$$

- It can be shown the quantum closure has to be of the form

$$P = L E L^{\dagger}$$

- where L is

$$L = \Upsilon_P^{\dagger} X^{1/2} \Upsilon_E$$

- all the arbitrary phases in the Φ’s can be redefined away.

- A quantum closure can be understood as a three-step relation:

$$P = Y_P^{\dagger}\left(X^{1/2}\left(Y_E E Y_E^{\dagger}\right)X^{1/2}\right)Y_P$$

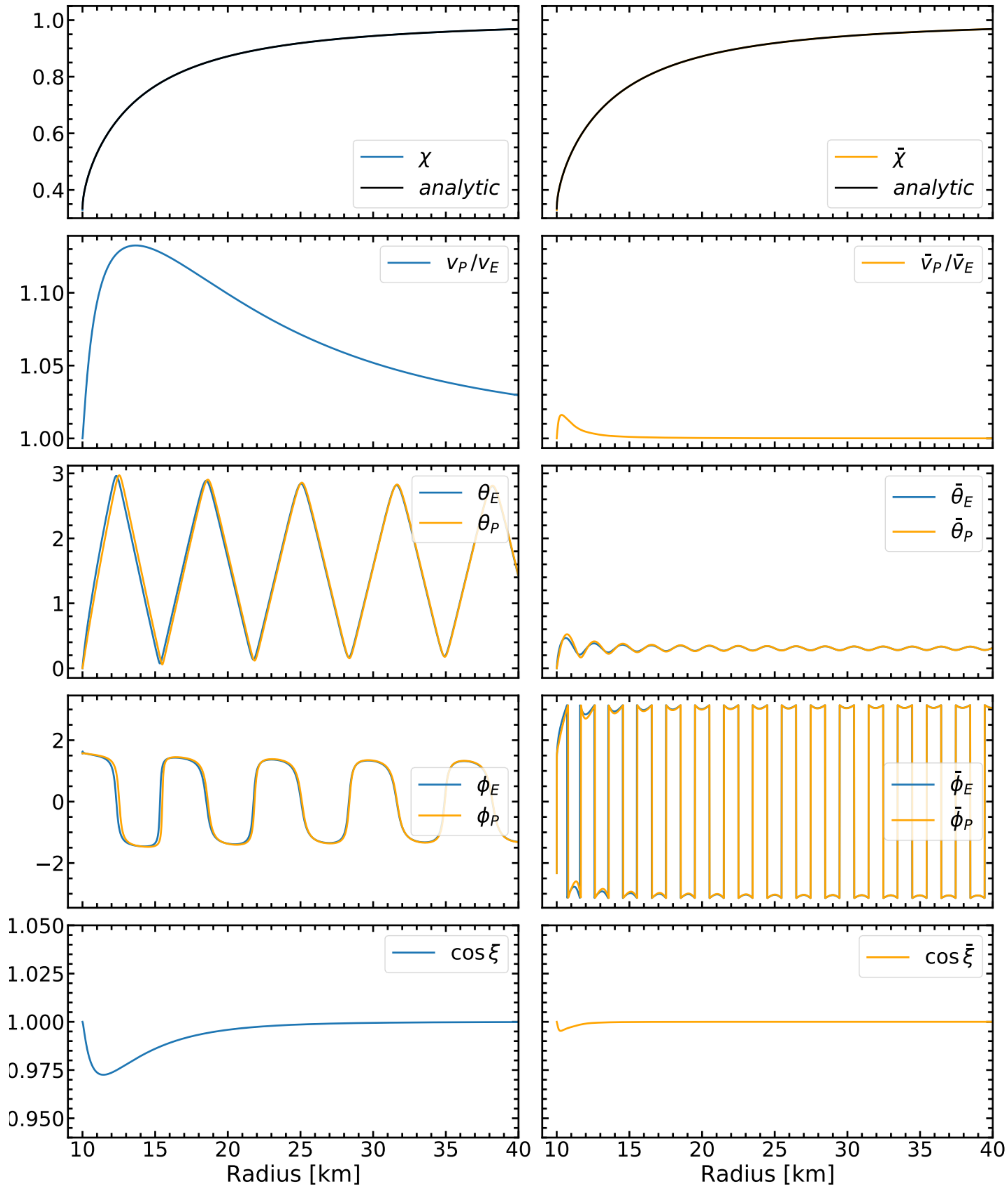
  - Y_E 'rotates' E to its eigenvalue matrix
  - $X^{1/2}$ rescales the eigenvalues
  - Y_P 'rotates' away from a diagonal matrix

Is the Quantum Closure Complete?

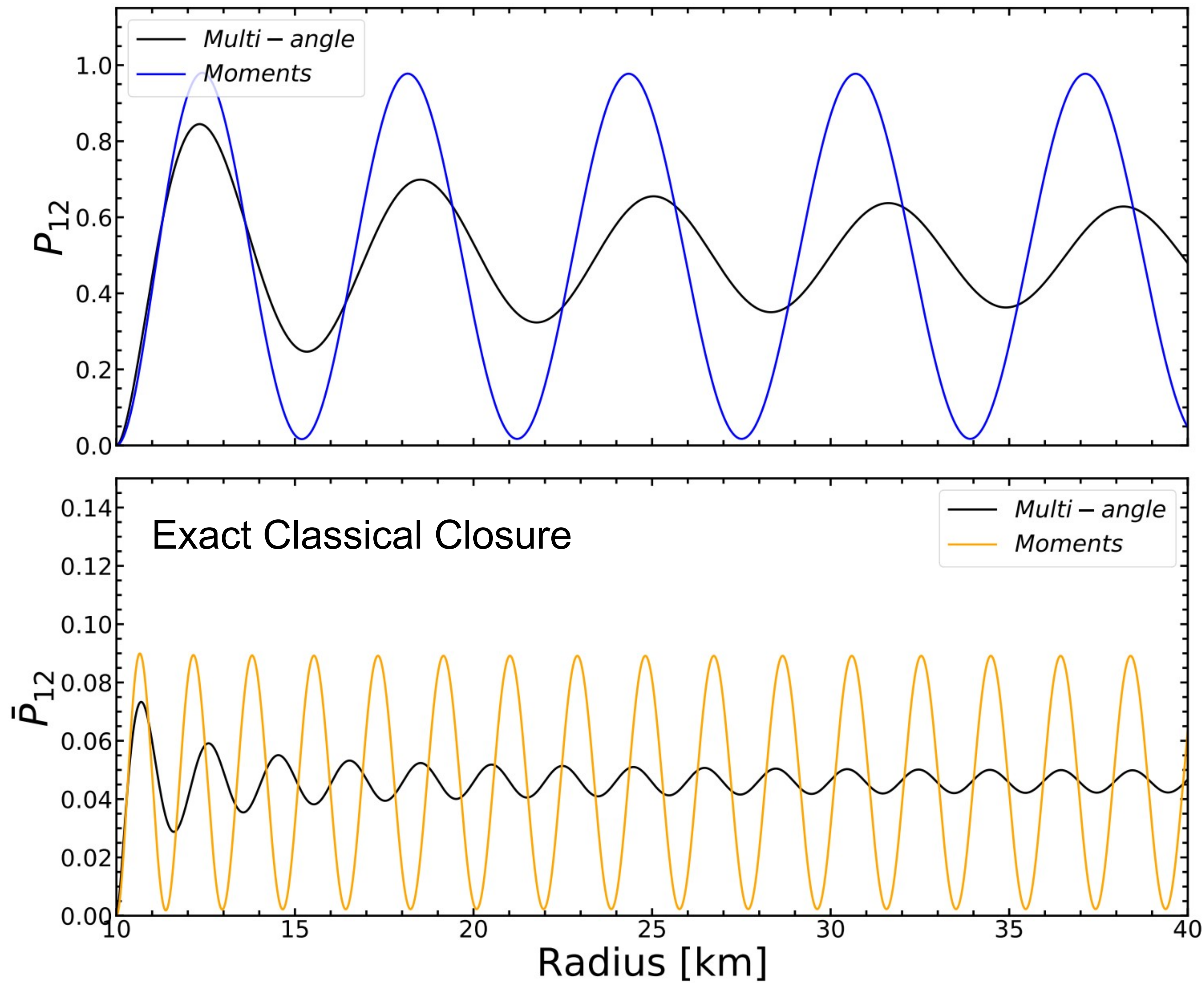
- If one uses the ‘correct’ closure in a moment calculation, the correct result must be recovered.

- Consider the MSW problem for neutrinos emitted from a spherical source (a lightbulb).
  - There is an exact solution for a single neutrino.
- We set the matter density and choose mixing parameters so that the neutrinos are on the MSW resonance.

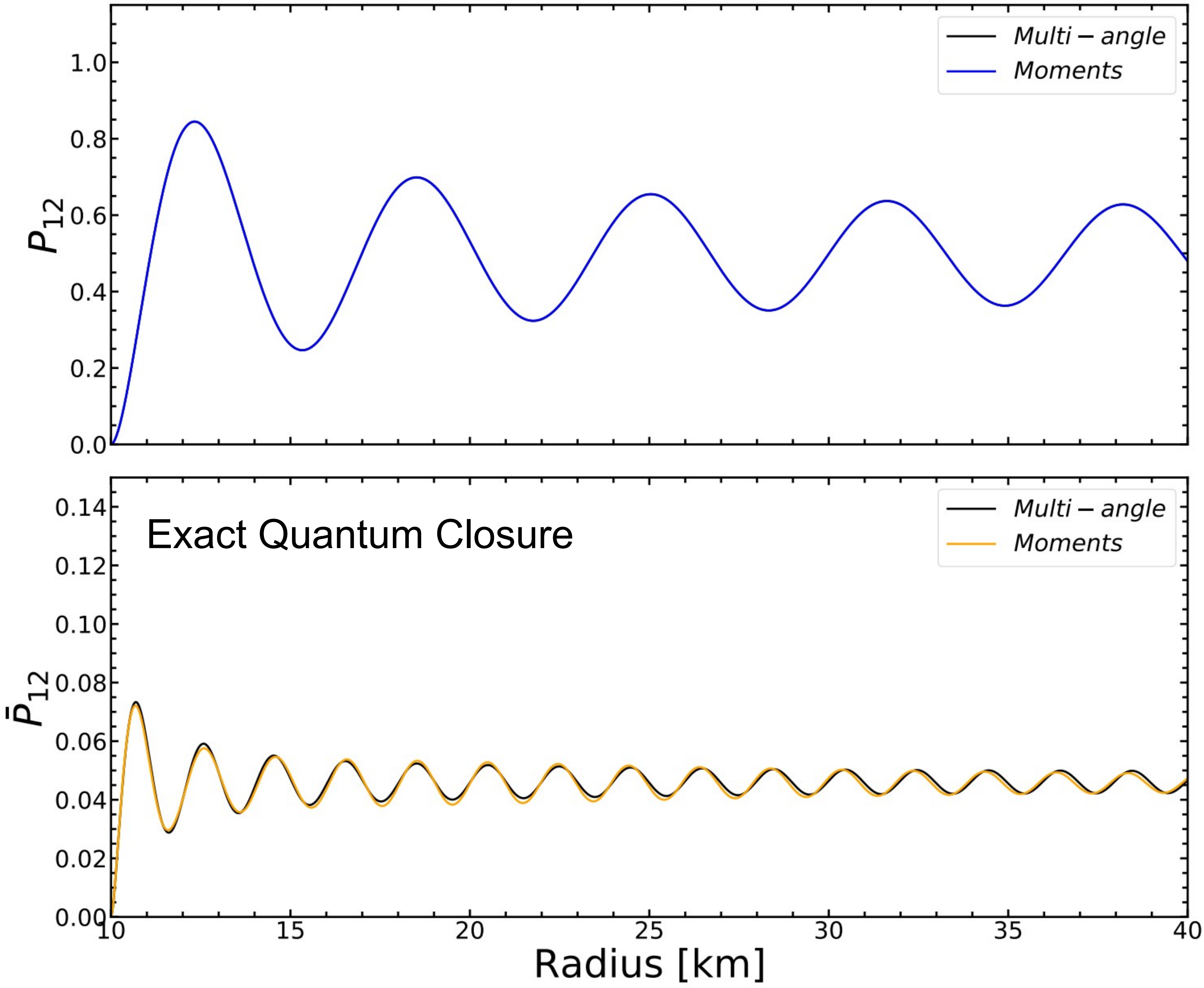
from the exact solution



- Now that we know the correct closure from the exact solution, we can do a moment calculation using this closure.



Multi − angle
Moments
P_{12}
1.0
0.8
0.6
0.4
0.2
0.0
Exact Classical Closure
Multi − angle
Moments
$\bar{P}_{12}$
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
10
15
20
25
30
35
40
Radius [km]



Multi − angle
Moments
P_{12}
Multi − angle
Moments
Exact Quantum Closure
$\bar{P}_{12}$
Radius [km]

Quantum Advection

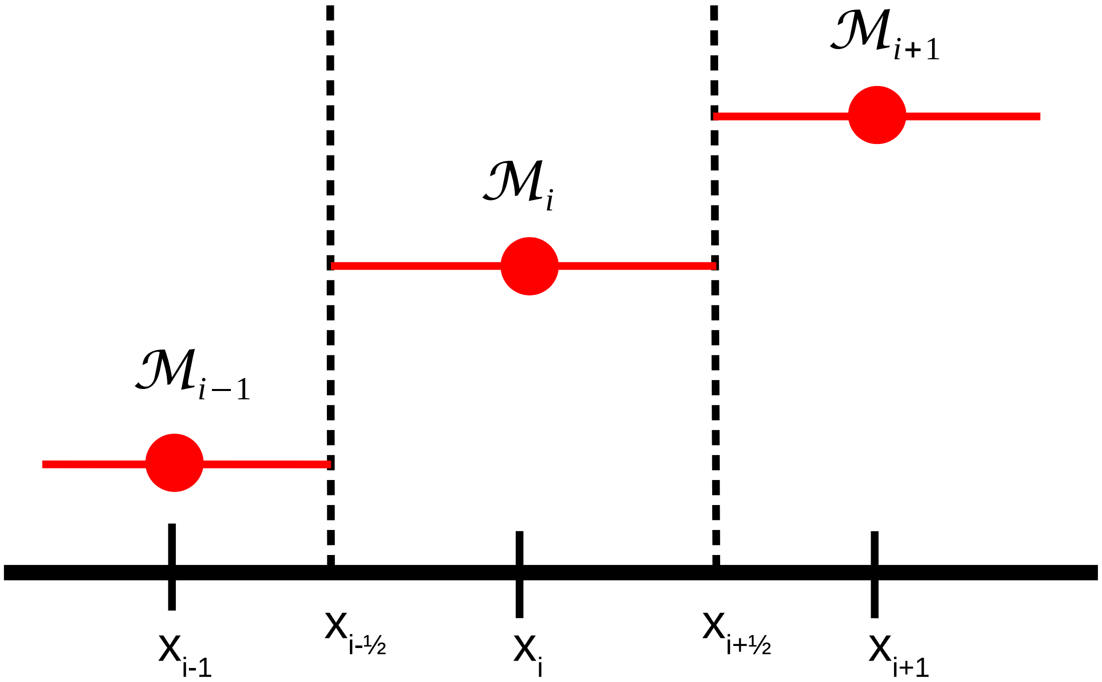
- Consider again the moment transport equations.

$$\frac{\partial E_q}{\partial t}+\frac{\partial F_q}{\partial r}+\frac{2F_q}{r}=-i[H_V+H_M+H_E,E_q]+i[H_F,F_q]$$

$$\frac{\partial F_q}{\partial t}+\frac{\partial P_q}{\partial r}+\frac{3P_q-E_q}{r}=-i[H_V+H_M+H_E,F_q]+i[H_F,P_q]$$

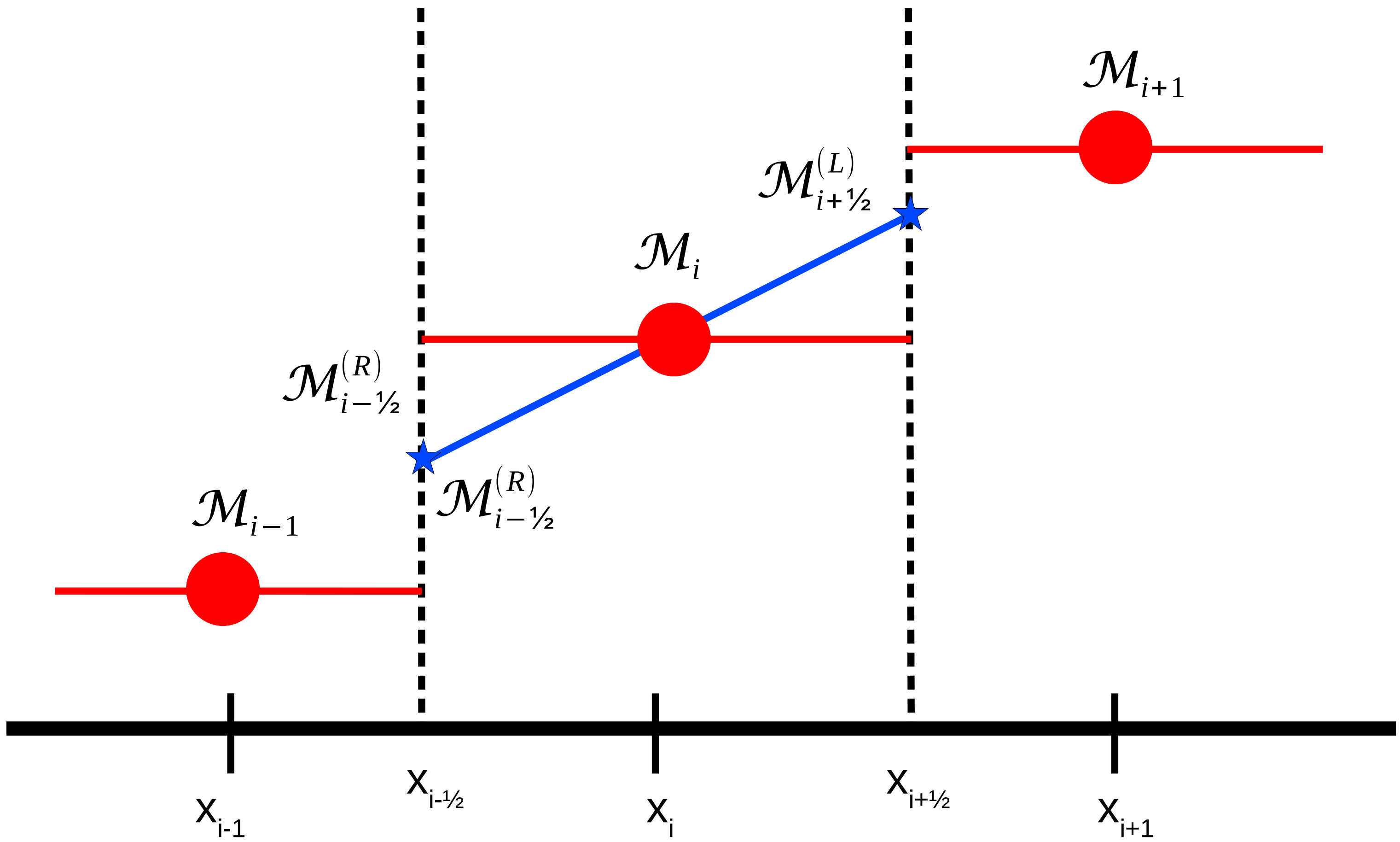
- To numerically solve these equations we move everything except the time derivatives to the right-hand side.
- We discretize the space into finite volumes and evolve the zone-averages of E and F within each grid zone.
- The zone averages ‘live’ at the cell centers.

- We end up with a set of ‘Riemann Problems’ for E and F: the initial value problem of the flow of some quantity across a discontinuity.

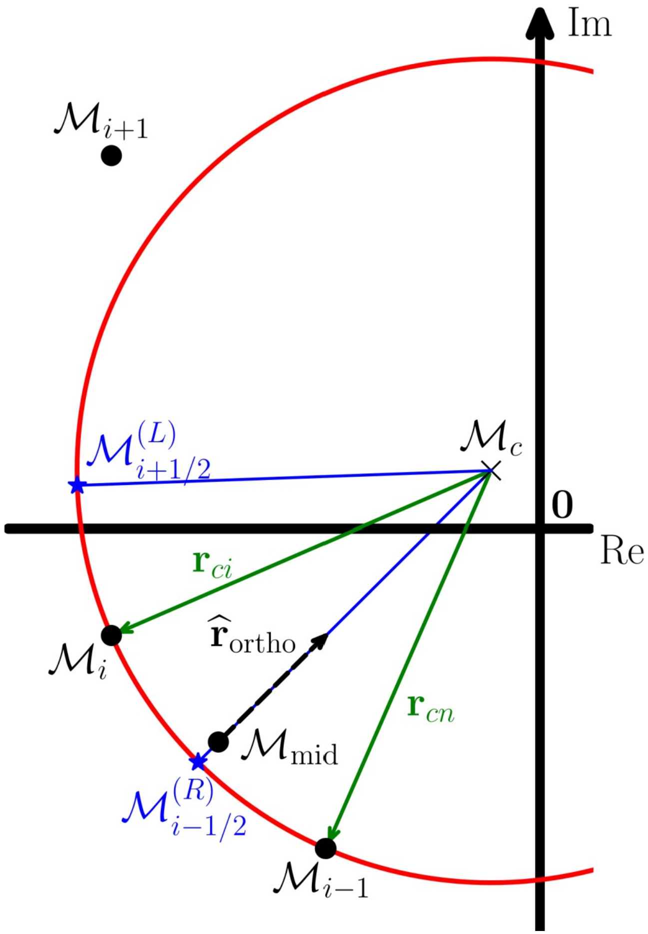


- There exist many algorithms – collectively known as ‘Riemann Solvers’ – for the classical version of this problem.
- For quantum moments, the off-diagonal elements – the coherences – are complex numbers.
- How do you build a Riemann Solver for quantum moments?

- Riemann Solvers have two parts: the ‘Reconstruction’ and the ‘Riemann Solution’.
- The Reconstruction step generates the values of the moments either side of the discontinuity from the values at the zone centers.



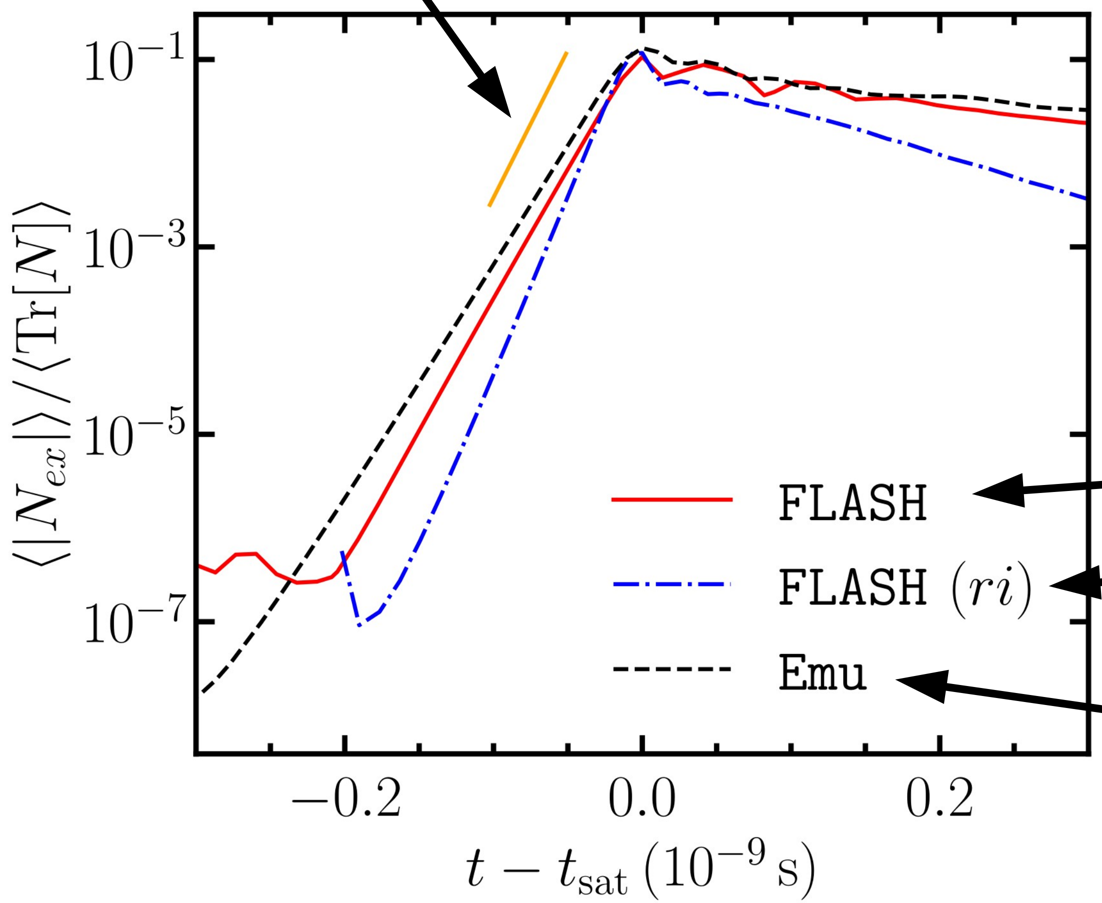
- We tried many ideas. The one that works most robustly is based on the ideas behind ‘minmod’ and uses interpolation using circular arcs in the complex (Argand) plane.



Are moment-based approaches any good?

- We have undertaken extensive testing of our version of FLASH with quantum moments vs EMU, a ‘multi-angle’ code.

Analytic prediction

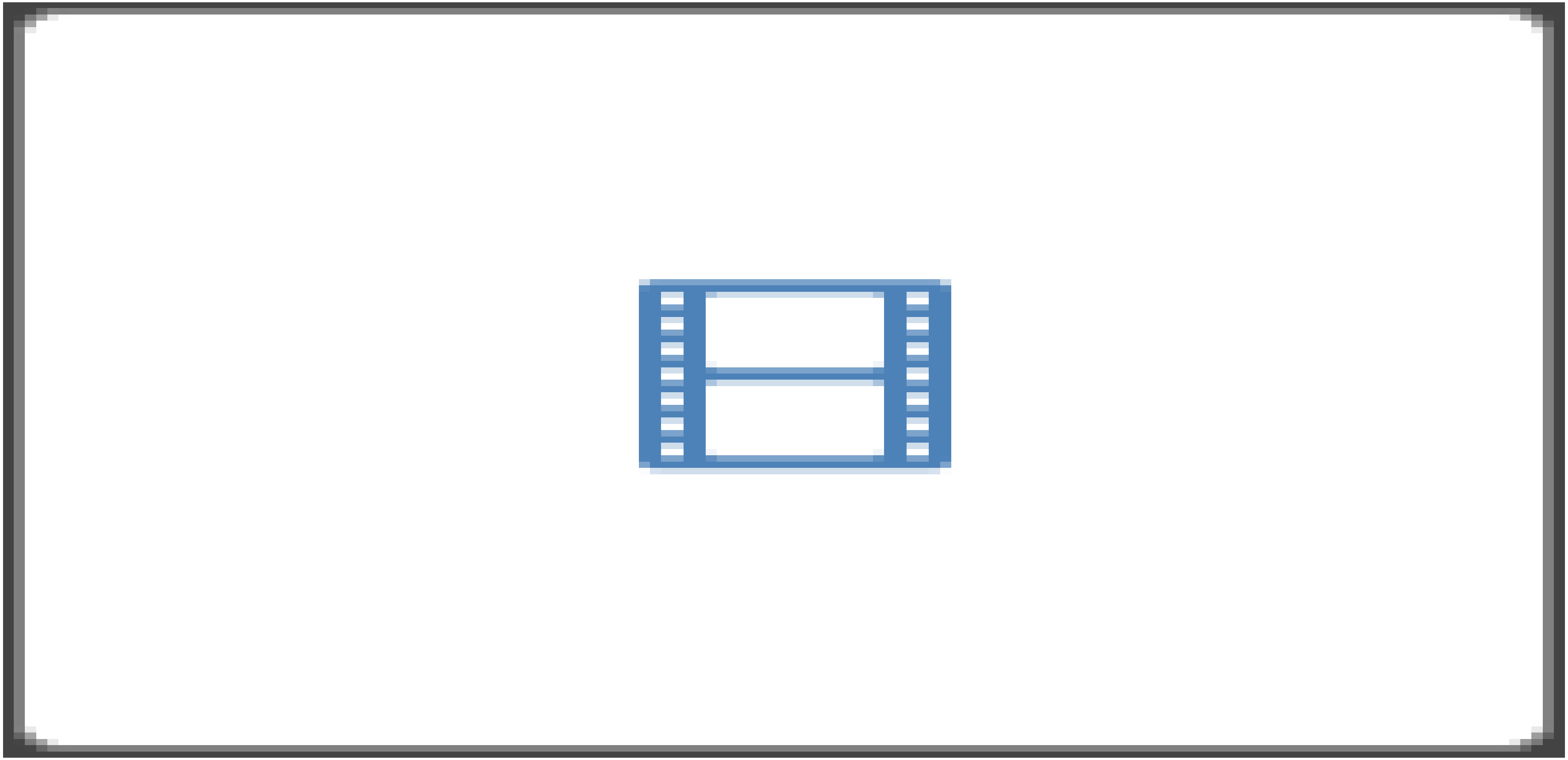


New moment code

Previous moment code

Multi-angle particle-in-cell code

Grohs et al, PRD **111** 083018 (2025)



- This animation is computed using our version of FLASH with quantum moments.
- It is the flavor transformation in a small volume using data taken from a binary neutron star merger by Foucart et al.

Summary

- Moments are a more efficient way of doing neutrino flavor transformation calculations.
- We know how to write a complete quantum closure, and if the correct closure is used, the results from moment calculations are exact.
- We have adapted classical Riemann solvers for quantum moments and they match results from ‘multi-angle’ codes.