The $H \to W^+W^-$ Decay Rate: From Lagrangian Density to Γ

(CDF Collaboration)

Jason Nett

(Dated: 6 July 2009)

I present here a detailed calculation of the $H \to W^+W^-$ decay rate.

Contents

LAGRANGIAN DENSITY TO INVARIANT AMPLITUDE

For a Higgs boson decay to two W bosons, the Lagrangian density comes from the Higgs sector of the standard model Lagrangian. The Z boson and photon terms can be excluded.

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{1}{2}\mu^2 H^2 + \frac{g^2 v^2}{4} W_\mu^\dagger W^\mu + \frac{g^2 v}{2} W_\mu^\dagger W^\mu H}_{\text{Higgs Sector}}$$
$$\underbrace{-\frac{1}{4}\sum_{i=1,2}(\partial_\mu W_{i\nu} - \partial_\nu W_{i\mu})(\partial^\mu W_i^\mu - \partial^\nu W_i^\mu)}_{\text{W boson kinetic terms}}$$

To calculate the invariant amplitude for the decay, I want the interaction Lagrangian.

$$\mathcal{L} = \frac{g^2 v}{2} W_\mu^\dagger W^\mu H$$

From this I need the interaction Hamiltonian density.

$$\mathcal{H}_I = \sum_{\text{fields}} \pi(x)\dot{\Phi}(x) - \mathcal{L}_I(x)$$

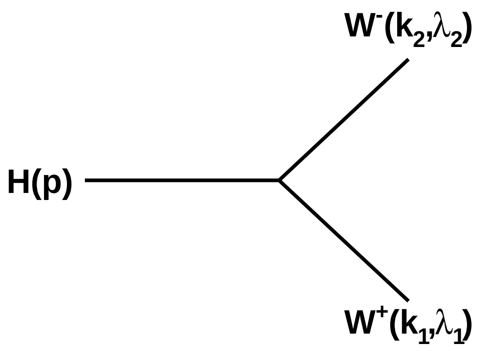

FIG. 1: $H \to W^+W^-$ Decay

where $\Phi(x)$ is a position-space field and $\pi(x)$ is its conjugate momentum field. However, in this case there are no time-derivatives of fields in the interaction Lagrangian density. So it is simply

$$\begin{aligned}\mathcal{H}_I(x) &= -\mathcal{L} \\ &= -\frac{g^2 v}{2} W_\mu^\dagger W^\mu H\end{aligned}$$

The scattering matrix for this interaction is

$$\langle k_1, k_2|S|p\rangle = \langle k_1, k_2|1|p\rangle + i\langle k_1, k_2|T|p\rangle$$

where we recall that the scattering matrix is defined as the time-evolution operator as $t \to \infty$.

$$\langle k_1, k_2|S|p\rangle = \lim_{t\to\infty} \langle k_1, k_2|e^{iH(2t)}|p\rangle$$

The interaction component here is what I want to calculate. From (4.90) Peskin,

$$i\langle k_1, k_2|T|p\rangle = \lim_{t\to\infty(1-i\varepsilon)} \langle k_1, k_2|T \exp\left[-i\int_{-t}^{t} dt' H_I(t')\right]|p\rangle$$

That exponential expands as (from (4.22) Peskin)

$$T\exp\left[-i\int_{-t}^{t} dt' H_I(t')\right] = 1 + (-i)\int_{-t}^{t} dt_1 H_I(t_1) + \frac{(-i)^2}{2!}\iint_{-t}^{t} dt_1 dt_2 T\left[H_I(t_1)H_I(t_2)\right] + \cdots$$

This scenario is just a tree-level decay–there are no loops to consider or propagators between two spacetime coordinates x_1 and x_2. Hence, let's consider only the contribution from the 1st order term. The interaction part of the scattering matrix becomes

$$i\langle k_1, k_2|T|p\rangle \cong \langle k_1, k_2|(-i)\int_{-t}^{t} dt_1 H_I(x_1)|p\rangle$$

where $H_I(x) = \int d^3x \mathcal{H}_I(x)$, and in the Hamiltonian I replaced the variable t with the full spacetime variable x because all components now come into play.

$$\begin{aligned}i\langle k_1, k_2|T|p\rangle &= (-i)\langle k_1, k_2|\int_{-t}^{t} dt_1 \int d^3x \mathcal{H}_I(x)|p\rangle \\ &= -i\langle k_1, k_2|\int d^4x \mathcal{H}_I(x)|p\rangle \\ &= -i\int d^4x \langle k_1, k_2|\mathcal{H}_I(x)|p\rangle \\ &= -i\int d^4x \langle k_1, k_2|\frac{-g^2 v}{2} W_\mu^\dagger W^\mu H|p\rangle \\ &= \frac{-ig^2 v}{2}\int d^4x \langle k_1, k_2|W_\mu^\dagger W^\mu H|p\rangle\end{aligned}$$

Assume the fields are now contracted with their state vectors.

If we go to my contractions section:

$$\langle k|W_\mu(x) = \langle 0|\varepsilon^*_\mu(k,\lambda)e^{ik\cdot x}$$
$$H|p\rangle = e^{-ip\cdot x}|0\rangle$$

Using these

$$\begin{aligned} i\langle k_1,k_2|T|p\rangle &= \frac{ig^2v}{2}\int d^4x\epsilon^*_\mu(k_1,\lambda_1)e^{ik_1\cdot x}\epsilon^{*^\mu}(k_2,\lambda_2)e^{ik_2\cdot x}e^{-ip\cdot x} \\ &= \frac{ig^2v}{2}\int d^4x\epsilon^*_\mu(k_1,\lambda_1)\epsilon^{*^\mu}(k_2,\lambda_2)e^{i(k_1+k_2-p)\cdot x} \\ &= \frac{ig^2v}{2}\epsilon^*_\mu(k_1,\lambda_1)\epsilon^{*^\mu}(k_2,\lambda_2)\delta^4(k_1+k_2-p)(2\pi)^4 \end{aligned}$$

Recall from (4.73) Peskin

$$\begin{aligned} i\langle k_1,k_2|T|p\rangle = (2\pi)^4\delta^4(k_1+k_2-p)\cdot i\mathcal{M}(p\to k_1,k_2) \\ \Rightarrow i\mathcal{M} = \frac{ig^2v}{2}\epsilon^*_\mu(k_1,\lambda_1)\epsilon^{*^\mu}(k_2,\lambda_2) \end{aligned}$$

INVARIANT AMPLITUDE TO DECAY RATE (Γ)

The decay rate from (4.83) Peskin is

$$\Gamma = \frac{1}{2m_H}\left[\frac{d^3k_1}{2E_1(2\pi)^3}\frac{d^3k_2}{2E_2(2\pi)^3}\right]\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2(2\pi)^4\delta^4(k_1+k_2-p)$$

So let's square the amplitude

$$\begin{aligned} i\mathcal{M} &= igm_W\epsilon^*_\mu(k_1,\lambda_1)\epsilon^{*^\mu}(k_2,\lambda_2) \\ |\mathcal{M}|^2 &= g^2m_W^2\left(\epsilon^*_\mu(k_1,\lambda_1)\epsilon_\nu(k_1,\lambda_1)\right)\left(\epsilon^{*^\mu}(k_2,\lambda_2)\epsilon^\nu(k_2,\lambda_2)\right) \end{aligned}$$

Now deal with the spin sum

$$\begin{aligned} \sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2 &= g^2m_W^2\sum_{\lambda_1,\lambda_2}\left(\epsilon^*_\mu(k_1,\lambda_1)\epsilon_\nu(k_1,\lambda_1)\right)\left(\epsilon^{*^\mu}(k_2,\lambda_2)\epsilon^\nu(k_2,\lambda_2)\right) \\ &= g^2m_W^2\left(-g_{\mu\nu}+\frac{k_{1\mu}k_{1\nu}}{m_W^2}\right)\left(-g^{\mu\nu}+\frac{k_2^\mu k_2^\nu}{m_W^2}\right) \\ &= g^2m_W^2\left(g_{\mu\nu}g^{\mu\nu}-g_{\mu\nu}\frac{k_2^\mu k_2^\nu}{m_W^2}-g^{\mu\nu}\frac{k_{1\mu}k_{1\nu}}{m_W^2}+\frac{k_{1\mu}k_{1\nu}k_2^\mu k_2^\nu}{m_W^2}\right) \\ &= g^2m_W^2\left(4-\frac{k_2^2}{m_W^2}-\frac{k_1^2}{m_W^2}+\frac{(k_1\cdot k_2)^2}{m_W^4}\right) \end{aligned}$$

Recall that in a reaction the 4-momentum squared is a relativistic invariant. Using this invariant, we may alternate among before and after the decay, and viewing from the lab frame or CM frame (or any other frame). In this case, let's try before and after decay entirely in the CM frame. This means $\vec{p}=0$, and for the W boson energy and momenta $E_1=E_2=E$, $\vec{k}_1=-\vec{k}_2\equiv\vec{k}$.

$$\Rightarrow \frac{k^2}{m_W^2} = \frac{E^2-|\vec{k}|^2}{m_W^2} = \frac{(m_W^2+|\vec{k}|^2)-|\vec{k}|^2}{m_W^2} = 1$$

Also,

$$\begin{aligned}
(m_H, 0)^2 &= (E_1 + E_2, \vec{k}_1 + \vec{k}_2 = 0)^2 \\
m_H^2 &= 4E_2^2 = 4(m_W^2 + |\vec{k}|^2) \\
m_H^2 &= 4m_W^2 + 4|\vec{k}|^2 \\
m_H^2 - 2m_W^2 &= 2(m_W^2 + 2|\vec{k}|^2) \\
\frac{m_H^2 - 2m_W^2}{2} &= k_1 \cdot k_2
\end{aligned}$$

where in the last line I used

$$\begin{aligned}
k_1 \cdot k_2 &= (E_1, \vec{k}_1) \cdot (E_2, \vec{k}_2) \\
&= (E, \vec{k}) \cdot (E_2, -\vec{k}) \\
&= E^2 + |\vec{k}|^2 \\
&= (m_W^2 + |\vec{k}|^2) + |\vec{k}|^2 \\
&= m_W^2 + 2|\vec{k}|^2
\end{aligned}$$

Now we may put these results back into the spin-summed invariant amplitude.

$$\begin{aligned}
\sum_{\lambda_1, \lambda_2} |\mathcal{M}|^2 &= g^2 m_W^2 \left(4 - 1 - 1 + \frac{(m_H^2 - 2m_W^2)^2}{4m_W^4}\right) \\
&= g^2 m_W^2 \left(2 + \frac{m_H^2 - 4m_W^2 m_H^2 + 4m_W^4}{4m_W^4}\right) \\
&= g^2 m_W^2 \left(2 + \frac{m_H^4}{4m_W^4} - \frac{m_H^2}{m_W^2} + 1\right) \\
&= g^2 m_W^2 \left(\frac{3m_H^4}{4m_W^4} \cdot \frac{4m_W^4}{m_H^4} + \frac{m_H^4}{4m_W^4} - \frac{m_H^4}{4m_W^4} \cdot \frac{4m_W^4}{m_H^4} \cdot \frac{m_H^2}{m_W^2}\right) \\
&= g^2 m_W^2 \frac{m_H^4}{4m_W^4} \left(1 + \frac{12m_W^4}{m_H^4} - \frac{4m_W^2}{m_H^2}\right) \\
&= \frac{g^2 m_H^4}{4m_W^4} \left(1 - \frac{4m_W^2}{m_H^2} + \frac{12m_W^4}{m_H^4}\right)
\end{aligned}$$

Put this into the decay rate.

$$\begin{aligned}
d\Gamma &= \frac{1}{2m_H} \left[\frac{d^3k_1}{2E_1(2\pi)^3} \frac{d^3k_2}{2E_2(2\pi)^3}\right] \frac{g^2 m_H^4}{4m_W^4} \left(1 - \frac{4m_W^2}{m_H^2} + \frac{12m_W^4}{m_H^4}\right) (2\pi)^4 \delta^4(k_1 + k_2 - p) \\
\Gamma &= \frac{g^2 m_H^3}{32(2\pi)^2 m_W^2} \left(1 - \frac{4m_W^2}{m_H^2} + \frac{12m_W^4}{m_H^4}\right) \int \frac{d^3\vec{k}_1}{E_1} \frac{d^3\vec{k}_2}{E_2} \delta(E_1 + E_2 - E_p) \delta^3(\vec{k}_1 + \vec{k}_2 - \vec{p})
\end{aligned}$$

In the CM frame, $E_p = m_H$ and $\vec{p} = 0$.

$$\begin{aligned}
\Gamma &= \frac{g^2 m_H^3}{32(2\pi)^2 m_W^2} \left(1 - \frac{4m_W^2}{m_H^2} + \frac{12m_W^4}{m_H^4}\right) \int \frac{d^3\vec{k}_1}{E_1} \frac{d^3\vec{k}_2}{E_2} \delta(E_1 + E_2 - m_H) \delta^3(\vec{k}_1 + \vec{k}_2) \\
\Gamma &= \frac{g^2 m_H^3}{32(2\pi)^2 m_W^2} \left(1 - \frac{4m_W^2}{m_H^2} + \frac{12m_W^4}{m_H^4}\right) \int \frac{d^3\vec{k}}{\sqrt{m_W^2 + |\vec{k}_1|^2}} \frac{d^3\vec{k}}{\sqrt{m_W^2 + |\vec{k}_2|^2}} \delta(E_1 + E_2 - m_H) \delta^3(\vec{k}_1 + \vec{k}_2)
\end{aligned}$$

Perform the k_2 integration. Because of $\delta^3(\vec{k}_1 + \vec{k}_2)$, this will just enforce $\vec{k}_1 = -\vec{k}_2 \Rightarrow |\vec{k}_1|^2 = |\vec{k}_2|^2$. Since we are dealing only with these momenta squared now, let's drop the index and just use $|\vec{k}|$.

$$\Gamma = \frac{g^2 m_H^3}{32(2\pi)^2 m_W^2}\left(1 - \frac{4m_W^2}{m_H^2} + \frac{12m_W^4}{m_H^4}\right)\int \frac{d^3\vec{k}}{m_W^2 + |\vec{k}|^2}\delta(E_1 + E_2 - m_H)$$

Express the remaining differential in spherical coordinates $d^3\vec{k} = |\vec{k}|^2 d|\vec{k}| \sin\theta d\theta d\phi$, where $\int \sin\theta d\theta d\phi = 4\pi$.

$$\Gamma = \frac{g^2 m_H^3}{32(2\pi)^2 m_W^2}\left(1 - \frac{4m_W^2}{m_H^2} + \frac{12m_W^4}{m_H^4}\right)\int \frac{|\vec{k}|^2 d|\vec{k}| \sin\theta d\theta d\phi}{m_W^2 + |\vec{k}|^2}\delta(E_1 + E_2 - m_H)$$

In the CM frame, $E_1 = E_2 = \sqrt{m_W^2 + |\vec{k}|^2}$.

$$\Gamma = \frac{g^2 m_H^3}{8(4\pi) m_W^2}\left(1 - \frac{4m_W^2}{m_H^2} + \frac{12m_W^4}{m_H^4}\right)\int d|\vec{k}| \frac{|\vec{k}|^2}{m_W^2 + |\vec{k}|^2}\delta\left(2\sqrt{m_W^2 + |\vec{k}|^2} - m_H\right)$$

We must take care here. The integral is over $|\vec{k}|$ but the argument of the δ-function has more than one zero, so there is an ambiguity of which value $|\vec{k}|$ should take from the integration. Fortunately, it is possible to expand the δ-function as follows:

$$\delta(f(x)) = \sum_j \frac{1}{f'\left(x_0^j\right)}\delta\left(x - x_0^j\right)$$

where j counts over the zeros of $f(x)$ and $f' = \frac{df}{dx}$. Let $f(k) = 2\sqrt{m_W^2 + |\vec{k}|^2} - m_H$ and find the zeros:

$$\begin{aligned}
m_H &= 2\sqrt{m_W^2 + |\vec{k}|^2} \\
\frac{m_H^2}{4} &= m_W^2 + |\vec{k}|^2 \\
\frac{1}{4}\left(m_H^2 - 2m_W^2\right) &= k^2 \\
\pm\frac{1}{2}\sqrt{m_H^2 - 2m_W^2} &= k
\end{aligned}$$

However, a negative momentum magnitude doesn't make sense so we only use the positive one.

$$\begin{aligned}
f'(k) &= \frac{2k}{\sqrt{m_W^2 + k^2}} \\
f'(k_0) &= \frac{\sqrt{m_H^2 - 4m_W^2}}{\sqrt{m_W^2 + \frac{1}{4}m_H^2 - m_W^2}} \\
f'(k_0) &= \frac{\sqrt{m_H^2 - 4m_W^2}}{\frac{1}{2}m_H^2} \\
f'(k_0) &= \frac{2}{m_H}\sqrt{m_H^2 - 4m_W^2} \\
\Rightarrow \delta(f(k)) &= -\frac{m_H}{2\sqrt{m_H^2 - 4m_W^2}}\delta\left(|\vec{k}| - \frac{1}{2}\sqrt{m_H^2 - 4m_W^2}\right)
\end{aligned}$$

Put this into the decay rate.

$$\begin{aligned}\Gamma &= \frac{g^2 m_H^3}{32\pi m_W^2}\left(1-\frac{4m_W^2}{m_H^2}+\frac{12m_W^4}{m_H^4}\right)\int d|\vec{k}|\frac{|\vec{k}|^2}{m_W^2+|\vec{k}|^2}\frac{m_H}{2\sqrt{m_H^2-4m_W^2}}\delta\left(|\vec{k}|-\frac{1}{2}\sqrt{m_H^2-4m_W^2}\right)\\
&= \frac{g^2 m_H^3}{32\pi m_W^2}\left(1-\frac{4m_W^2}{m_H^2}+\frac{12m_W^4}{m_H^4}\right)\frac{\frac{1}{4}(m_H^2-4m_W^2)}{m_W^2+\frac{1}{4}(m_H^2-4m_W^2)}\frac{m_H}{2\sqrt{m_H^2-4m_W^2}}\\
&= \frac{g^2 m_H^3}{32\pi m_W^2}\left(1-\frac{4m_W^2}{m_H^2}+\frac{12m_W^4}{m_H^4}\right)\frac{\frac{1}{4}(m_H^2-4m_W^2)}{\frac{1}{4}m_H^2}\frac{m_H}{2\sqrt{m_H^2-4m_W^2}}\\
&= \frac{g^2 m_H^3}{32\pi m_W^2}\left(1-\frac{4m_W^2}{m_H^2}+\frac{12m_W^4}{m_H^4}\right)\frac{\sqrt{m_H^2-4m_W^2}}{2m_H}\\
&= \frac{g^2 m_H^3}{64\pi m_W^2}\left(1-\frac{4m_W^2}{m_H^2}+\frac{12m_W^4}{m_H^4}\right)\sqrt{1-\frac{4m_W^2}{m_H^2}}\\
&= \frac{G_F m_H^3}{8\sqrt{2}\pi}\left(1-\frac{4m_W^2}{m_H^2}+\frac{12m_W^4}{m_H^4}\right)\sqrt{1-\frac{4m_W^2}{m_H^2}}\end{aligned}$$

where $G_F = \frac{\sqrt{2}g^2}{8m_W^2} \Rightarrow \frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$.