A CDM Model Based On An Exact Solution Of The Einstein-Sterile Neutrino-Acceleron Field Equations

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We couple a neutral scalar field and a Majorana fermion field to Einstein gravity represented by the Robertson-Walker metric and find a class of exact cosmological solutions.

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Introduction

Recent progress both in cosmology and in neutrino physics is very fast. In cosmology, initial evidence for the acceleration of the universe from high red-shift supernova observations was subsequently verified by the precise measurements of the cosmic microwave background radiation. These observations and measurements discovered that there is a significant "dark-energy" component of the universe. In a parallel development solar, atmospheric, and reactor neutrino experiments firmly established that neutrinos are massive and that they mix. Mass-varying neutrinos were introduced in part as an attempt to explain the origin of the cosmological dark-energy density and why its magnitude is apparently coincidental with that of the measured neutrino mass splittings [1]. Such mass-varying neutrinos can behave as a negative pressure fluid leading to the acceleration of the universe.

Models for the dark energy in which the energy density of the scalar field approximates Einstein's cosmological constant were studied in detail. It can be shown that in the mass-varying neutrino scenario dark energy is also equivalent to having a cosmological constant [2]. It has been argued that mass-varying neutrino models contain an instability when neutrinos become non-relativistic and a stable neutrino-varying mass model is indistinguishable from a cosmological constant [3]. Mass-varying neutrino models without acceleron like scalar fields were also proposed [4]. However, an analysis of the cosmic microwave background radiation anisotropies and large scale structure implies some evidence for coupling between neutrinos and a scalar field [5].

Quantization of a Fermi field, coupled to the Robertson-Walker metric, had been worked out some time ago [6]. Motivated by the recent studies of mass-varying neutrinos, we explored [7] if exact solutions exist when a neutral scalar field (representing the acceleron of the mass-varying neutrino models) as well as a Majorana fermion field (representing the neutrino) couples to gravity via Robertson-Walker metric. We show that such solutions indeed exist.

The Cosmological Model

We solve coupled Einstein-acceleron-Majorana neutrino field equations derived by a variational principle from the Lagrangian density

$$\mathcal{L} = \frac{1}{2\kappa}\mathcal{R}*1 - \frac{1}{2}d\phi\wedge *d\phi + V(\phi)*1 + i\bar{\psi}(\gamma\wedge *\nabla)\psi - iM(\phi)\bar{\psi}\psi *1 \quad (1)$$

where ψ is the 4-component (Majorana spinor) sterile neutrino field and ϕ is the (real,scalar) acceleron field; $V(\phi)$ the acceleron potential function, $M(\phi)$ the varying-mass of the neutral fermion to be determined. $\kappa = 8\pi G$ is the gravitational coupling constant (in natural units such that $c = 1 = \hbar$). The space-time geometry is given in terms of a metric tensor g of Lorentzian signature $-+++$ and its unique Levi-Civita connection ∇. $\mathcal{R}$ is the corresponding curvature scalar and $*1$ is the oriented volume element.

We use a set of real γ-matrices (Majorana realization) $\{\gamma_a\}$ that satisfy

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab} I. \tag{2}$$

Reality means ${\gamma^*}_a = \gamma_a$. Hence we have ${\gamma^\dagger}_0 = {\gamma^T}_0 = -\gamma_0$ and ${\gamma^\dagger}_i = {\gamma^T}_i = \gamma_i$. The conjugate spinor field is

$$\bar{\psi} = \psi^\dagger C, \tag{3}$$

where the charge conjugation matrix $C = \gamma_0$. With our conventions the spinor field ψ is self-conjugate if and only if it is real. That is

$$\psi^C \equiv C \bar{\psi}^T = \psi^*. \tag{4}$$

We look for exact cosmological solutions of the coupled system of field equations.

We use the Robertson-Walker metric (spatially flat, $k = 0$)

$$g = -dt \otimes dt + R^2(t)(dx \otimes dx + dy \otimes dy + dz \otimes dz) \tag{5}$$

in terms of the cosmic time t and isotropic coordinates x^i : (x, y, z). $R(t) \geq 0$ is the expansion function. We further let

$$\phi = \phi(t) \tag{6}$$

and

$$\psi = (h_1(t) + h_2(t)\gamma_0)\xi \quad , \quad \bar{\psi} = \bar{\xi}(h_1(t) - h_2(t)\gamma_0) \tag{7}$$

where $h_1(t), h_2(t)$ are functions to be determined and ξ is a constant Majorana 4-spinor. We substitute these in the variational field equations and reduce them to a set of ordinary differential equations:

$$\frac{3}{\kappa}\left(\frac{\dot{R}}{R}\right)^2 - \frac{\dot{\phi}^2}{2} + V(\phi) = \mathcal{N} M(\phi)({h_1}^2 + {h_2}^2) \tag{8}$$

$$\begin{aligned} \frac{2}{\kappa}\frac{\ddot{R}}{R} + \frac{1}{\kappa}\left(\frac{\dot{R}}{R}\right)^2 + \frac{\dot{\phi}^2}{2} + V(\phi) &= \mathcal{N}(h_2\dot{h}_1 - h_1\dot{h}_2) \\ &\quad + \mathcal{N} M(\phi)({h_1}^2 + {h_2}^2) \end{aligned} \tag{9}$$

$$\ddot{\phi} + 3\frac{\dot{R}}{R}\dot{\phi} - \frac{dV}{d\phi} = -\mathcal{N}(\frac{dM}{d\phi})({h_1}^2 + {h_2}^2) \tag{10}$$

$$\dot{h}_1 + \frac{3\dot{R}}{2R}h_1 + M(\phi)h_2 = 0 \tag{11}$$

$$\dot{h}_2 + \frac{3\dot{R}}{2R}h_2 - M(\phi)h_1 = 0 \tag{12}$$

where

$$i(\bar{\xi}\xi) \equiv \mathcal{N}. \tag{13}$$

Remark: 1 If we differentiate (8) and simplify by using (10), (11), (12) and (8) again, we obtain precisely (9). Thus, out of the five equations above, only four of them are independent. On the other hand we have four functions to solve for. Therefore the system is well-determined.

Remark: 2 If we use (11) and (12) for the derivatives of h_1 and h_2 and substitute in (9), the right hand side vanishes. This means that in our simple model the neutrino pressure is zero.

Remark: 3 (11) and (12) implies that

$$R^3({h_1}^2 + {h_2}^2) = C \tag{14}$$

is a constant of motion. We designate $n = \mathcal{N}C$ which may be called the neutrino number density.

An Exact Solution

We only consider the simple case of a constant potential $V(\phi) = V_0$. First we solve the resulting equations,

$$
\begin{aligned}
\frac{2}{\kappa}\frac{\ddot{R}}{R} + \frac{1}{\kappa}\left(\frac{\dot{R}}{R}\right)^2 + \frac{\dot{\phi}^2}{2} + V_0 &= 0, && (15)\\
\frac{3}{\kappa}\left(\frac{\dot{R}}{R}\right)^2 - \frac{\dot{\phi}^2}{2} + V_0 - \frac{n}{R^3}M(\phi) &= 0, && (16)\\
\ddot{\phi} + 3\frac{\dot{R}}{R}\dot{\phi} + \frac{n}{R^3}\frac{dM}{d\phi} &= 0, && (17)
\end{aligned}
$$

by assuming the existence of solutions that satisfy

$$R(t) = R_0 e^{\alpha\phi(t)} \tag{18}$$

where both R_0 and α are constants to be chosen later.

We then solve (15) and get

$$\dot{\phi}(t) = \sqrt{\frac{2\kappa V_0}{\kappa + 6\alpha^2}} \cot\left(\sqrt{\frac{\kappa\Lambda}{8\alpha^2}(\kappa + 6\alpha^2)}\, t\right). \tag{19}$$

Inserting this expression into (16) determines the functional form of $M(\phi)$. In order to get a closed expression for it, we first write ϕ as a function of t:

$$\phi(t) = \phi_0 - \frac{2\alpha}{\kappa + 6\alpha^2} \ln\left|\csc^2\left(\sqrt{\frac{\kappa V_0}{8\alpha^2}(\kappa + 6\alpha^2)}\, t\right)\right|. \tag{20}$$

Here ϕ_0 is an integration constant. Taking the exponential of both sides and using trigonometric identities we find

$$\cot^2\left(\sqrt{\frac{\kappa V_0}{8\alpha^2}(\kappa + 6\alpha^2)}\, t\right) = e^{-\frac{\kappa + 6\alpha^2}{2\alpha}(\phi - \phi_0)} - 1. \tag{21}$$

Substituting this in (16), we solve for the mass function as a function of ϕ:

$$nM(\phi) = \frac{2\kappa V_0 {R_0}^3}{\kappa + 6\alpha^2}\left\{e^{3\alpha\phi} + \frac{6\alpha^2 - \kappa}{2\kappa} e^{\frac{\kappa+6\alpha^2}{2\alpha}\phi_0} e^{-\frac{\kappa}{2\alpha}\phi}\right\}. \tag{22}$$

The mass-function may be given a better parametrization in terms of its critical value such that

$$\left.\frac{dM}{d\phi}\right|_{\phi=\phi_c} = 0. \tag{23}$$

The critical value, ϕ_C, may be determined from the relation

$$e^{\frac{\kappa+6\alpha^2}{2\alpha}\phi_0} = \frac{12\alpha^2}{6\alpha^2 - \kappa} e^{\frac{\kappa+6\alpha^2}{2\alpha}\phi_C}. \tag{24}$$

Therefore we may write

$$nM(\phi) = \frac{2\kappa V_0 {R_0}^3}{\kappa + 6\alpha^2} e^{3\alpha\phi_C} \left\{ e^{3\alpha(\phi-\phi_C)} + \frac{6\alpha^2}{\kappa} e^{-\frac{\kappa}{2\alpha}(\phi-\phi_C)} \right\}. \tag{25}$$

The minimum mass value is

$$nM(\phi_C) = 2V_0 {R_0}^3 e^{3\alpha\phi_C}. \tag{26}$$

Differentiating with respect to ϕ we find

$$n\frac{dM}{d\phi} e^{-3\alpha\phi} = \frac{6\alpha\kappa V_0 {R_0}^3}{\kappa + 6\alpha^2} \left\{ 1 - e^{-\frac{\kappa+6\alpha^2}{2\alpha}(\phi-\phi_C)} \right\}. \tag{27}$$

It is now possible to substitute this expression into (17) and verify that it is identically satisfied. This is a consistency check.

To complete the solution, we go to the Dirac system:

$$
\begin{aligned}
\dot{h}_1 + \frac{3\dot{R}}{2R}h_1 + M(\phi)h_2 &= 0, \\
\dot{h}_2 + \frac{3\dot{R}}{2R}h_2 - M(\phi)h_1 &= 0.
\end{aligned}
$$

These can be integrated formally and the general solution reads

$$
\begin{aligned}
h_1(t) &= e^{-\frac{3}{2}\alpha\phi}\{A\cos\Omega(t) + B\sin\Omega(t)\} \\
h_2(t) &= e^{-\frac{3}{2}\alpha\phi}\{A\sin\Omega(t) - B\cos\Omega(t)\}
\end{aligned}
$$

where A, B are the initial values and

$$
\Omega(t) = \int_0^t M(t')dt'. \tag{28}
$$

The mass is a complicated function of t, so we will leave the solution implicit in terms of $\Omega(t)$.

In fact the Majorana spinor field for the solution is of the form

$$\psi = e^{-\frac{3}{2}\alpha\phi+\gamma_0\Omega}(A - B\gamma_o)\xi. \tag{29}$$

To conclude, we work out the expansion function as a function of t

$$R(t) = R_0 e^{\alpha\phi_C}\left|\frac{12\alpha^2}{6\alpha^2-\kappa}\sin^2\left(\sqrt{\frac{\kappa V_0}{8\alpha^2}(\kappa+6\alpha^2)t}\right)\right|^{\frac{2\alpha^2}{\kappa+6\alpha^2}}. \tag{30}$$

It is not difficult to determine the following observable quantities; the Hubble parameter:

$$H(t) \equiv \frac{\dot{R}}{R} = \sqrt{\frac{2\kappa\alpha^2 V_0}{\kappa + 6\alpha^2}} \cot\left(\sqrt{\frac{\kappa V_0}{8\alpha^2}(\kappa + 6\alpha^2)}t\right), \tag{31}$$

and the acceleration parameter:

$$\frac{R\ddot{R}}{\dot{R}^2} = 1 - \frac{\kappa + 6\alpha^2}{4\alpha^2} \csc^2\left(\sqrt{\frac{\kappa V_0}{8\alpha^2}(\kappa + 6\alpha^2)}t\right). \tag{32}$$

In the limit where $\kappa \ll \alpha^2$ and $V_0 \sim 1$, to the first order of approximation we would have

$$R(t) \sim t^{\frac{2}{3}}, \tag{33}$$

i.e. a matter-dominated Friedmann universe.

We next consider the case of a cosmological constant: $V_0 = -\Lambda$. In this case Eqs. (15), (16) and (17) can be considered representing a generalization of the Λ_{CDM} model. The solution of Eq. (15) now becomes

$$\dot{\phi}(t) = \sqrt{\frac{2\kappa\Lambda}{\kappa + 6\alpha^2}} \coth\left(\sqrt{\frac{\kappa\Lambda}{8\alpha^2}(\kappa + 6\alpha^2)}t\right), \tag{34}$$

yielding

$$\phi(t) = \phi_0 + \frac{2\alpha}{\kappa + 6\alpha^2} \ln\left|\sinh^2\left(\sqrt{\frac{\kappa\Lambda}{8\alpha^2}(\kappa + 6\alpha^2)}t\right)\right|, \tag{35}$$

and

$$\coth^2\left(\sqrt{\frac{\kappa\Lambda}{8\alpha^2}(\kappa + 6\alpha^2)}t\right) = e^{-\frac{\kappa+6\alpha^2}{2\alpha}(\phi-\phi_0)} + 1. \tag{36}$$

In this case we find the mass function to be

$$nM(\phi) = \frac{2\kappa\Lambda {R_0}^3}{\kappa + 6\alpha^2}\left\{-e^{3\alpha\phi} + \frac{6\alpha^2 - \kappa}{2\kappa} e^{\frac{\kappa+6\alpha^2}{2\alpha}\phi_0} e^{-\frac{\kappa}{2\alpha}\phi}\right\}. \tag{37}$$

Rest of the solutions of the Dirac system is unchanged. The expansion function now has the form

$$R(t) = R_0 e^{\alpha\phi_0}\left[\sinh\left(\sqrt{\frac{\kappa\Lambda}{8\alpha^2}(\kappa + 6\alpha^2)}t\right)\right]^{\frac{4\alpha^2}{\kappa+6\alpha^2}}. \tag{38}$$

We also note the forms of the Hubble parameter

$$H(t) = \sqrt{\frac{2\kappa\alpha^2\Lambda}{\kappa + 6\alpha^2}} \coth\left(\sqrt{\frac{\kappa\Lambda}{8\alpha^2}(\kappa + 6\alpha^2)}t\right), \tag{39}$$

and the acceleration parameter:

$$\frac{R\ddot{R}}{\dot{R}^2} = 1 - \frac{\kappa + 6\alpha^2}{4\alpha^2} \cosh^{-2}\left(\sqrt{\frac{\kappa\Lambda}{8\alpha^2}(\kappa + 6\alpha^2)}t\right). \tag{40}$$

In the limit $\kappa \ll \alpha^2$ and $\Lambda \sim 1$, Eq. (38) again yields the matter-dominated Friedmann universe, $R(t) \sim t^{2/3}$. In this limit we get the Hubble parameter to be

$$H(t) \sim \frac{1}{t_\Lambda} \coth(t/t_\Lambda), \tag{41}$$

and the expansion factor to be

$$R(t) \sim \sinh^{2/3}\left(\frac{t}{t_\Lambda}\right), \tag{42}$$

where $t_\Lambda^{-1} = (\sqrt{3\kappa\Lambda})/2$. These values of the Hubble parameter and the expansion factor are those obtained in the flat Λ_{CDM} model.

Conclusion

We have given exact solutions to a cosmological model where a neutral scalar field and a Majorana fermion field are coupled to gravity represented by the Robertson-Walker metric. We were motivated by the mass-varying neutrino models where the scalar field could represent an acceleron field and the Majorana fermion field a sterile neutrino. However, our results are valid in a broader context, generalizing the earlier solution of Isham and Nelson. The constants R_0 and α of the expansion function and the critical value ϕ_c of the scalar field are parameters that can be adjusted to construct different phenomenological models.

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