Neutrino Mass in the Standard Model

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Cosmic neutrinos decouple from the Big Bang plasma at a temperature around 2 MeV. At that time they have a thermal Fermi-Dirac distribution. As the universe expands, their density and temperature red-shift, leading to

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma = 1.95\mathrm{K}; \qquad n_{\nu_i} = n_{\overline{\nu}_i} = \frac{3}{22} n_\gamma = \frac{56}{\mathrm{cm}^3}$$

$$\eta_\nu = \frac{n_\nu - n_{\overline{\nu}}}{n_\gamma} \simeq \eta_b = \frac{n_b - n_{\overline{b}}}{n_\gamma} \simeq 10^{-10}$$

where T_γ and n_γ are the measured temperature and number density of CMB photons. Neutrinos density is enhanced by gravitational clustering [Singh, Ma; Ringwald, Wong].
Due to large mixing, the flavor composition is equilibrated. All three mass eigenstates have equal densities. [Lunardini, Smirnov]

What scales do we know about?

$p_F = \sqrt[3]{3\pi^2 n}$	2.34×10^{-4} eV	per flavor/anti
T_ν	1.68×10^{-4} eV	
$\sqrt{\Delta m^2_{21}}$	8.75×10^{-3} eV	
$\sqrt{\Delta m^2_{31}}$	4.90×10^{-2} eV	

- Because $m_\nu \simeq p_F$ we must ask: what is the contribution of p_F to m_ν?
- Vacuum field theory is the approximation $p_F = 0$.
- While $p_F < \Delta m$, the number density of neutrinos is the *average* number density throughout the universe. We should expect that the density is enhanced in gravitational potentials such as our solar system and galaxy.

Why this is hard

- Finite density field theory is constructed with only one number operator, the chemical potential μ
- Majorana fermions violate μ conservation (Only the Majorana mass operator could be allowed within the SM)
- This medium has both particles and anti-particles, and their numbers are separately conserved.

  $\Rightarrow$ What is the number operator?
- This is finite temperature, finite density, out of equilibrium quantum field theory. $\Rightarrow$ New tools must be developed.

Why this is easy

- The system is “just” a Free Fermi Gas.

The "vacuum" background state of our universe is described by a set of creation and annihilation operators in momentum space (columns)

$$a^\dagger_{p_i}|n_1,\cdots,n_i,\cdots,n_N\rangle = \sqrt{n_i+1}\,|n_1,\cdots,n_i+1,\cdots,n_N\rangle\,;$$
$$a_{p_i}|n_1,\cdots,n_i,\cdots,n_N\rangle = \sqrt{n_i}\,|n_1,\cdots,n_i-1,\cdots,n_N\rangle\,.$$

A Weyl fermion has two creation operators, $a^\dagger_{+p}, a^\dagger_{-p}$

$$\{a^\dagger_{+p},a_{+q}\} = \{a^\dagger_{-p},a_{-q}\} = \delta_{pq},$$
$$\{a^\dagger_{\pm p},a^\dagger_{\pm q}\} = \{a_{\pm p},a_{\pm q}\} = \{a^\dagger_{+p},a_{-q}\} = 0$$

The subscript $\pm$ labels the two helicities (aka particle/anti-particle). The full vacuum is

$$|\Psi\rangle = \sum_i^{4^N}\alpha_i\left|\begin{array}{cccc} n^1_+ & n^2_+ & n^3_+ & \cdots \\ n^1_- & n^2_- & n^3_- & \cdots \end{array}\right\rangle$$

Fermions are never alone

Given N neutrinos (or anti-neutrinos), the basis Fock states of the system are Slater determinants

$$\left| \begin{array}{cccc} n^1_+ & n^2_+ & n^3_+ & \cdots \\ n^1_- & n^2_- & n^3_- & \cdots \end{array} \right\rangle = \left| \begin{array}{ccc} \psi_1(p_1) & \cdots & \psi_N(p_1) \\ \vdots & & \vdots \\ \psi_1(p_N) & \cdots & \psi_N(p_N). \end{array} \right|$$

where $\psi(p)$ is a plane wave.

- If I make a probe neutrino of momentum p_1, there are $N-1$ ways that the "probe" neutrino is not the one carrying momentum p_1!
- The intuitive picture of a single neutrino propagating is a *commuting fermion* intuition.

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$$\left| \begin{array}{cccc} n_+^1 & n_+^2 & n_+^3 & \cdots \\ n_-^1 & n_-^2 & n_-^3 & \cdots \end{array} \right\rangle = \left| \begin{array}{ccc} \psi_1(p_1) & \cdots & \psi_N(p_1) \\ \vdots & & \vdots \\ \psi_1(p_N) & \cdots & \psi_N(p_N). \end{array} \right|$$

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In order to construct a *local* field theory that includes this medium, we want to sum all contributions to a particular momentum mode.

$$|\Psi\rangle = \sum_{j,p} \Bigg[A^j_{0p} \left| \begin{matrix} n^1_+ & \cdots & n^{p-1}_+ & 0 & n^{p+1}_+ \cdots \\ n^1_- & \cdots & n^{p-1}_- & 0 & n^{p+1}_- \cdots \end{matrix} \right\rangle$$
$$+A^j_{+p} \left| \begin{matrix} n^1_+ & \cdots & n^{p-1}_+ & 1 & n^{p+1}_+ \cdots \\ n^1_- & \cdots & n^{p-1}_- & 0 & n^{p+1}_- \cdots \end{matrix} \right\rangle$$
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$$+A^j_{2p} \left| \begin{matrix} n^1_+ & \cdots & n^{p-1}_+ & 1 & n^{p+1}_+ \cdots \\ n^1_- & \cdots & n^{p-1}_- & 1 & n^{p+1}_- \cdots \end{matrix} \right\rangle \Bigg]$$

p sums momentum modes and j sums the configurations of all other momenta besides p.

This medium can be described by a series of expectation values describing its properties. The quadratic and quartic ones are:

$$N_0 = \langle a_+ a_- a_-^\dagger a_+^\dagger \rangle = \sum_{j,p} |A_{0p}^j|^2; \qquad N_+ = \langle a_+^\dagger a_- a_-^\dagger a_+ \rangle = \sum_{j,p} |A_{+p}^j|^2;$$

$$N_- = \langle a_-^\dagger a_+ a_+^\dagger a_- \rangle = \sum_{j,p} |A_{-p}^j|^2; \qquad N_2 = \langle a_+^\dagger a_-^\dagger a_- a_+ \rangle = \sum_{j,p} |A_{2p}^j|^2.$$

$$N_m = \langle a_+^\dagger a_- \rangle = \sum_{j,p} A_{-p}^j A_{+p}^{j*}; \qquad N_s = \langle a_+^\dagger a_-^\dagger \rangle = \sum_{j,p} A_{0p}^j A_{2p}^{*j}.$$

- N_0, N_+, N_-, N_2 count the number of occupied modes in different configurations.
- N_m is a complex Majorana mass
- N_s is a complex Fermi surface mixing operator

Single Particle Effective Field Theory

- What we must do is constructing the *single particle* effective theory: We replace the medium with expectation values that a single particle excitation would see.
- To build this field theory, we can use Lagrange multipliers to fix all the local expectation values of the medium. For example, add to the Lagrangian

$$\lambda \left(a_+^\dagger a_- - \langle a_+^\dagger a_- \rangle \right)$$

where λ is a Lagrange multiplier. The expectation values themselves are constants, and non-dynamical, so can be dropped. The resulting Lagrangian is a *Mean Field Theory*.
- Because the medium contains *both particles and anti-particles*, we can both create and annihilate the medium. e.g. $a_\pm |\Psi\rangle \neq 0$ instead of $a_\pm |0\rangle = 0$.

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Our Field

The field we must use is a Nambu-Gor'kov spinor. That is, place all the creation and annihilation operators into one spinor.

$$\psi = \begin{pmatrix} a_+^\dagger \\ a_- \\ a_-^\dagger \\ -a_+ \end{pmatrix}$$

Notice that this is also the standard definition of a Majorana spinor.

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What is a Majorana Mass?

- The Majorana mass operators are $a_+^\dagger a_-$ and $a_-^\dagger a_+$. They correspond to the field theory operators $\overline{\psi}\psi$ and $\overline{\psi}\gamma^5\psi$. It is complex. Its magnitude is the kinematic mass and phase is called the Majorana phase.
- The states responsible for this operator getting an expectation value are *superpositions of singly-occupied modes*. For instance with only one mode,

$$|\Psi\rangle = \alpha \left| \begin{array}{cc} 1 & \cdots \\ 0 & \cdots \end{array} \right\rangle + \beta \left| \begin{array}{cc} 0 & \cdots \\ 1 & \cdots \end{array} \right\rangle$$

has

$$\langle a_+^\dagger a_- \rangle = \alpha^* \beta$$

How to create a $\nu - \overline{\nu}$ superposition

- It is natural to expect a superposition, consider the process $\nu\bar{\nu} \to \nu\bar{\nu}$ mediated by a Z boson. The final state can have $\nu(p)$ and $\bar{\nu}(k)$ or $\nu(k)$ and $\bar{\nu}(p)$.
- If we do not observe which way the neutrino and anti-neutrino went, for each momentum mode the final state has a superposition

$$A\nu(p) + B\bar{\nu}(p)$$

where A and B are the amplitudes for the process to emit a neutrino in the direction p or an anti-neutrino in the direction p.
- *Every* scattering interaction in a thermal bath creates a superposition.

Majorana mass corresponds to an entropy

- In a multi-particle neutrino bath, the presence of a Majorana mass indicates that the bath has *spin-entropy*.

$$S = -\rho \ln \rho = N \ln 2$$

- Maximixing the spin-entropy corresponds to a state that is an equal superposition of all 2^N possible states choosing of which momentum mode is neutrino and which are antineutrino.
- The cosmological relic neutrinos came from a thermal bath, and should have maximal entropy.
- The two bilinear operators N_m and N_s are related to the mixing entropy of the singly and doubly occupied modes.
- This kind of entropy is a mixing entropy and is *non-extensive*

The Field Theory of a Free Fermi Gas

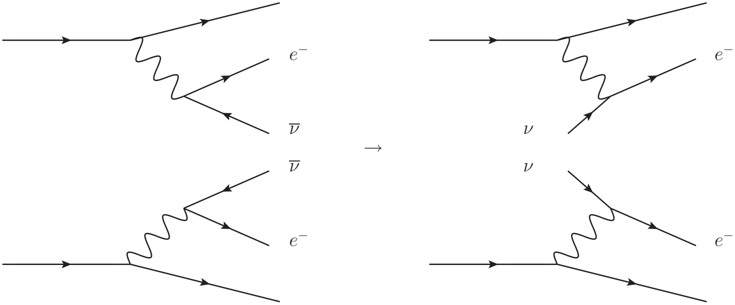
$$\begin{aligned}
\mathcal{L} = i\overline{\psi}\partial\!\!\!/\psi &+ \mu\overline{\psi}\gamma^0\gamma^5\psi \\
&+ m\cos\alpha_m\overline{\psi}\psi + m\sin\alpha_m\overline{\psi}\gamma^5\psi \\
&+ s\cos\alpha_s\overline{\psi}\gamma^1\gamma^5\psi + s\sin\alpha_s\overline{\psi}\gamma^2\gamma^5\psi + \eta\overline{\psi}\gamma^3\gamma^5\psi \\
&+ \frac{1}{4M^2}\overline{\psi}(\gamma^1 + i\gamma^2)\gamma^5\psi\overline{\psi}(\gamma^1 - i\gamma^2)\gamma^5\psi \\
&+ \frac{1}{4M^2}\overline{\psi}(\gamma^1 - i\gamma^2)\gamma^5\psi\overline{\psi}(\gamma^1 + i\gamma^2)\gamma^5\psi \\
&+ \frac{1}{4M^2}\overline{\psi}P_L\psi\overline{\psi}P_R\psi + \frac{1}{4M^2}\overline{\psi}P_R\psi\overline{\psi}P_L\psi.
\end{aligned}$$

- μ chemical potential; m Majorana mass (spin-entropy); α_m Majorana phase; s Fermi surface mixing; α_s Fermi surface mixing phase; M Pauli blocking “interaction”

Comments on the Field Theory of a Free Fermi Gas

- We can choose the quartic Lagrange Multiplier M such that all 4 quartics are identical. Any residual ends up in $\overline{\psi}\gamma^3\gamma^5\psi \to a_+^\dagger a_+ + a_-^\dagger a_-$ and $\overline{\psi}\gamma^0\gamma^5\psi \to a_+^\dagger a_+ - a_-^\dagger a_-$.
- We can “bosonize” the quartic using a Hubbard-Stratanovich transformation $\Rightarrow$ two complex auxiliary fields with mass M.
- One loop self-energy in Imaginary Time formalism gives $m(T)$.
- Standard textbooks, real time formalism, imaginary time formalism are missing all these order parameters except μ.
- This has implications far beyond neutrinos: finite temperature QCD?

Neutrinoless double beta decay



- The $0\nu\beta\beta$ process *directly measures* the expectation value $\langle a_+^\dagger a_- \rangle$.
- Consider the standard $2\nu\beta\beta$ process, now flip both neutrinos to the initial state.

What can be predicted

Calculations unfortunately not finished...
Rewriting the textbooks on the Free Fermi Gas was required...

- $m(T)$
- Hierarchy predictable
- $\sin^2\theta_{13}$ fixed by other mixing angles, mass differences.
- smallest mass fixed by structure of mass matrix (and is $\mathcal{O}(\Delta m_{12})$)
- Mass measurement gives local number density of neutrinos (which might be measurable in another way)

Conclusions

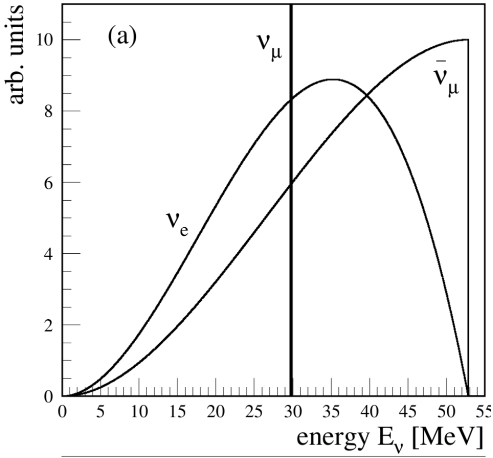
- The Standard Model can have neutrino mass, as a finite-density effect.
- Such a mass is *Majorana*
- A Majorana spinor is a Nambu-Gor'kov (finite density) spinor
- Neutrinoless double beta decay absorbs a pair of neutrinos from the relic background.
- Neutrino flavor and fermion number are *conserved* by the Standard Model, but violated by the medium.
- The conserved number operator for a free fermi gas is a *quartic* operator

Because the neutrino mass is an environmental quantity, *we can change the environment*.

- In the vicinity of the sun, neutrino “mass” should be increased due to larger gravitational potential. Is this compatible with solar mixing experiments?
- In a supernova, $p_F \sim$MeV. What can be said here?
- Since muon decay creates both a neutrino and anti-neutrino, near an intense muon source the effective mass of the neutrino is modified! Could this explain LSND? Future experiments a la KARMEN at a neutron spallation source (OscSNS?).

Muon Spectrum from a stopped pion source

- ν_μ from $\pi^+ \to \mu^+ \nu_\mu$
- ν_e and $\overline{\nu}_\mu$ from $\mu^+ \to e^+ \nu_e \overline{\nu}_\mu$
- ν_μ from $\pi^+ \to \mu^+ \nu_\mu$
- $\alpha\, \nu_e + \beta\, \overline{\nu}_\mu$ from muon decay
- e.g. a ν_e observed at 40 MeV is a superposition with a $\overline{\nu}_\mu$ with 50% probability for each.
- The relative height of the ν_e and $\overline{\nu}_\mu$ curves tells you the relative size of α and β.
- (Slightly oversimplified by ignoring angular correlations)



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