TriMinimal Parametrization of Neutrino Mixing Matrix

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Some neutrino Flavor physics

Besides energy and direction, cosmic quanta carry intrinsic information.

For cosmic-rays, it is A and Z;

For photons, it is spin polarization;

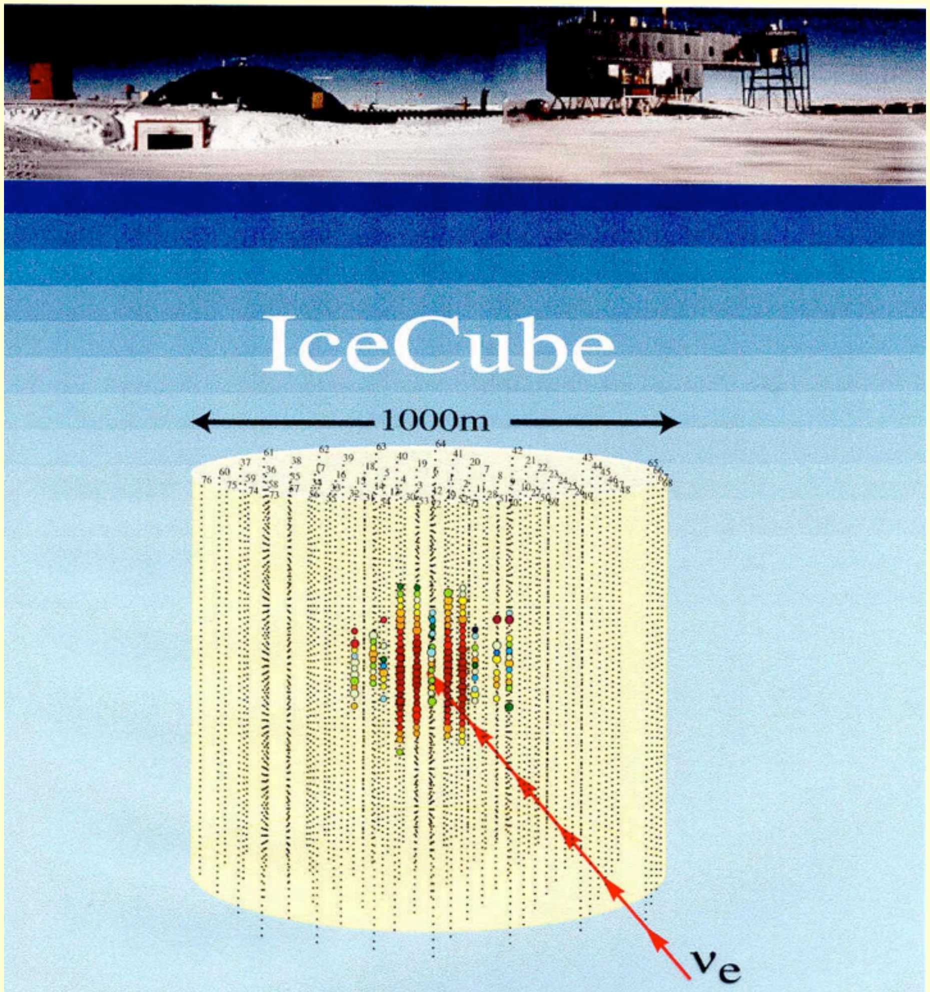
For neutrinos, it is flavor:

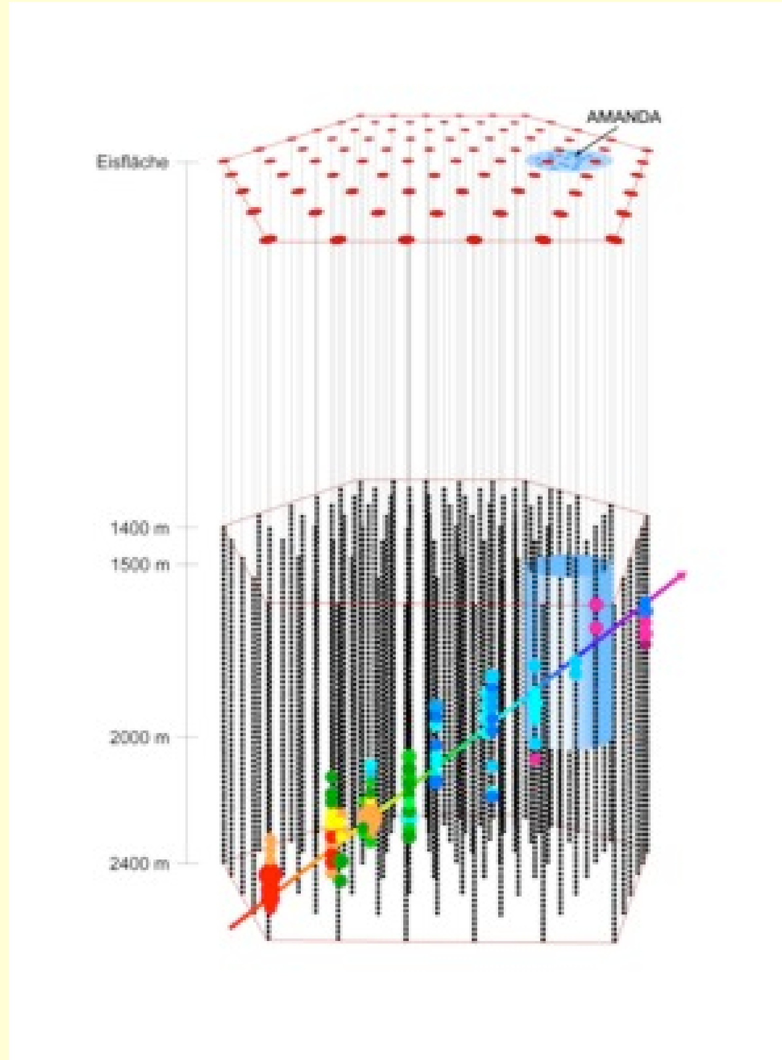
electron-neutrino (which showers)
muon neutrino (whose CC tracks)
tau neutrino (which showers below a PeV, tracks above)

Moreover, the flavors mix in a calculable/known way,
which means the flavors oscillate in an L/E-dependent way,
enabling:

Flavor "Interferometry" over Cosmic baselines !!

IceCube - 2010 completion, 30% 2007

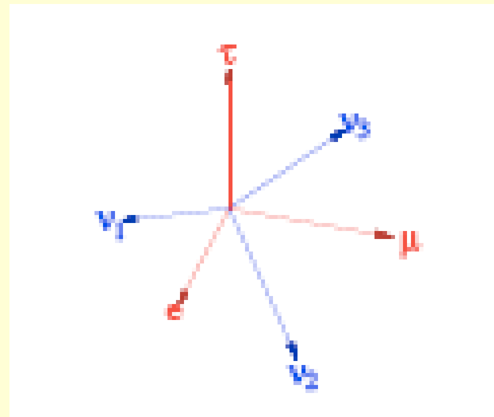




What we know:

$$
\begin{aligned}
U &= R_{32}(\theta_{32})\, U_\delta^\dagger\, R_{13}(\theta_{13})\, U_\delta\, R_{21}(\theta_{21}) \qquad (1)\\
&= \begin{pmatrix} c_{21}c_{13} & s_{21}c_{13} & s_{13}e^{-i\delta} \\ -s_{21}c_{32} - c_{21}s_{32}s_{13}e^{i\delta} & c_{21}c_{32} - s_{21}s_{32}s_{13}e^{i\delta} & s_{32}c_{13} \\ s_{21}s_{32} - c_{21}c_{32}s_{13}e^{i\delta} & -c_{21}s_{32} - s_{21}c_{32}s_{13}e^{i\delta} & c_{32}c_{13} \end{pmatrix}
\end{aligned}
$$

where $R_{jk}(\theta_{jk})$ describes a rotation in the jk-plane through angle θ_{jk}, $U_\delta = \mathrm{diag}(e^{i\delta/2}, 1, e^{-i\delta/2})$, and $s_{jk} = \sin\theta_{jk}$, $c_{jk} = \cos\theta_{jk}$. We have omitted two additional Majorana phases, as they do not affect neutrino oscillations.



from Strumia and Vissani

flavor fits to data give the following (1σ) and 3σ ranges for the PDG mixing angles [3]:

$$
\begin{aligned}
\sin^2\theta_{21} &= 0.32\,(\pm 0.02)\,{}^{+0.08}_{-0.06}\,, \\
\sin^2\theta_{32} &= 0.45\,(\pm 0.07)\,{}^{+0.20}_{-0.13}\,, \\
\sin^2\theta_{13} &< (0.02)\;0.050\,.
\end{aligned}
$$

Figure 2.4: *Possible neutrino spectra: (a) normal (b) inverted.*

The central values of these inferred mixing angles are quite consistent with the TriBiMaximal values [4], given

TriMinimal Parametrization of Neutrino Mixing

Pakvasa, Rodejohann, TJW [PRL (2008)]

$$U_{\rm TBM} = R_{32}\left(\frac{\pi}{4}\right) R_{21}\left(\sin^{-1}\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ 1 & -\sqrt{2} & \sqrt{3} \end{pmatrix}$$

The TriMinimal parametrization is given by

$$\begin{aligned} U_{\rm TMin} &= R_{32}\left(\frac{\pi}{4}\right) U_\epsilon(\epsilon_{32};\epsilon_{13},\delta;\epsilon_{21})\, R_{21}\left(\sin^{-1}\frac{1}{\sqrt{3}}\right) \\ &= \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \frac{U_\epsilon}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & 1 & 0 \\ -1 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}, \end{aligned}$$

$$\text{with } U_\epsilon = R_{32}(\epsilon_{32})\, U_\delta^\dagger\, R_{13}(\epsilon_{13})\, U_\delta\, R_{21}(\epsilon_{21}) \qquad (8)$$

chosen to have just the form of the PDG parametrization. And just as in the PDG parametrization, U_ϵ is unitary, and therefore so is TriMinimal $U_{\rm TMin}$. The simplicity of Eq. (8) is a fortuitous result of the fact that it is the middle rotation angle (θ_{13}) in the PDG parametrization that is set identically to zero in the TriBiMaximal scheme.

the small ϵ_{jk} are obtained from the (1σ) 3σ ranges of the large oscillation angles in Eq. (6). The allowed ranges are:

$$-0.08\ (-0.04) \le \epsilon_{21} \le (0.01)\ 0.07\,, \qquad (13)$$

$$-0.18\ (-0.10) \le \epsilon_{32} \le (0.04)\ 0.15\,, \qquad (14)$$

$$|\epsilon_{13}| \le (0.14)\ 0.23\,, \qquad (15)$$

while the CP-invariant lies in the range $|J_{\rm CP}| \le (0.03)\ 0.05$.

TriMinimal U

$$
\begin{aligned}
U_{\rm TMin} = U_{\rm TBM} \; &- \frac{\epsilon_{21}}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & 0 \\ -\sqrt{2} & -1 & 0 \end{pmatrix} - \frac{\epsilon_{32}}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ -1 & \sqrt{2} & -\sqrt{3} \\ -1 & \sqrt{2} & \sqrt{3} \end{pmatrix} - \frac{\epsilon_{13}}{\sqrt{6}} \begin{pmatrix} 0 & 0 & -\sqrt{6}\, e^{-i\delta} \\ \sqrt{2}\, e^{i\delta} & e^{i\delta} & 0 \\ \sqrt{2}\, e^{i\delta} & e^{i\delta} & 0 \end{pmatrix} \\
&- \frac{\epsilon_{21}^2}{2\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & 0 \\ 1 & -\sqrt{2} & 0 \end{pmatrix} - \frac{\epsilon_{32}^2}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ 1 & -\sqrt{2} & \sqrt{3} \end{pmatrix} - \frac{\epsilon_{13}^2}{2\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} \end{pmatrix} \\
&- \frac{\epsilon_{21}\,\epsilon_{32}}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & -1 & 0 \\ -\sqrt{2} & -1 & 0 \end{pmatrix} - \frac{\epsilon_{21}\,\epsilon_{13}\, e^{i\delta}}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ -1 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & 0 \end{pmatrix} - \frac{\epsilon_{32}\,\epsilon_{13}\, e^{i\delta}}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 1 & 0 \\ -\sqrt{2} & -1 & 0 \end{pmatrix} + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

- Unitary to order (eps)^2
- General, with three (mini-) angles and a phase

Long-Distance QM evolves to Classical Probabilities

of classical probabilities, defined by $(\underline{U})_{\alpha j} \equiv |U_{\alpha j}|^2$. The full matrix of squared elements, through order $\mathcal{O}(\epsilon^2)$, is

$$\underline{U}_{\rm TMin} = \frac{1}{6}\left\{ \begin{pmatrix} 4 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} - E_1 \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} - E_2 \begin{pmatrix} 2 & -2 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} - 2\,\epsilon_{32} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & -3 \\ -1 & -2 & 3 \end{pmatrix} - \epsilon_{13}^2 \begin{pmatrix} 4 & 2 & -6 \\ -2 & -1 & 3 \\ -2 & -1 & 3 \end{pmatrix} \right\} \tag{17}$$

where $E_1 = 2\sqrt{2}\,\epsilon_{13}\cos\delta + 2\,\epsilon_{21}\,(\epsilon_{13}\cos\delta - 2\sqrt{2}\,\epsilon_{32})$, and $E_2 = 2\sqrt{2}\,\epsilon_{21} + \epsilon_{21}^2$. That there are four independent terms in (17) reflects the fact that there are four independent moduli in the neutrino mixing matrix [9].

Some useful results follow immediately from this matrix. For example, in models where neutrinos are unstable, only the lightest neutrino mass-eigenstate arrives at Earth from cosmically-distant sources [10]. Flavor ratios at Earth for the normal mass-hierarchy are $\underline{U}_{e1} : \underline{U}_{\mu 1} : \underline{U}_{\tau 1}$, and for the inverted mass-hierarchy are $\underline{U}_{e3} : \underline{U}_{\mu 3} : \underline{U}_{\tau 3}$. These ratios may be read off directly from the 1st and 3rd columns of Eq. (17). As another example, emanating solar neutrinos are nearly pure ν_2 mass states [11]; consequently, their flavor ratios at Earth are mainly given by the 2nd column of (17).

Figure 2.4: *Possible neutrino spectra: (a) normal (b) inverted.*

Decohering the PMNS/tribimaximal neutrino-mixing matrix

$$\text{Decoherence} \rightarrow \; |U_{\alpha j}|^2 = \frac{1}{6} \begin{array}{cc} & \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ & \left(\begin{array}{ccc} 4 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{array} \right) \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \end{array}$$

E.g., pion decay chain neutrinos evolve to

$2/3\ \nu_\mu + 1/3\ \nu_e \rightarrow \nu_1{:}\nu_2{:}\nu_3 = 1{:}1{:}1$

$\rightarrow \nu_e{:}\nu_\mu{:}\nu_\tau = 1{:}1{:}1$

Democracy Broken:

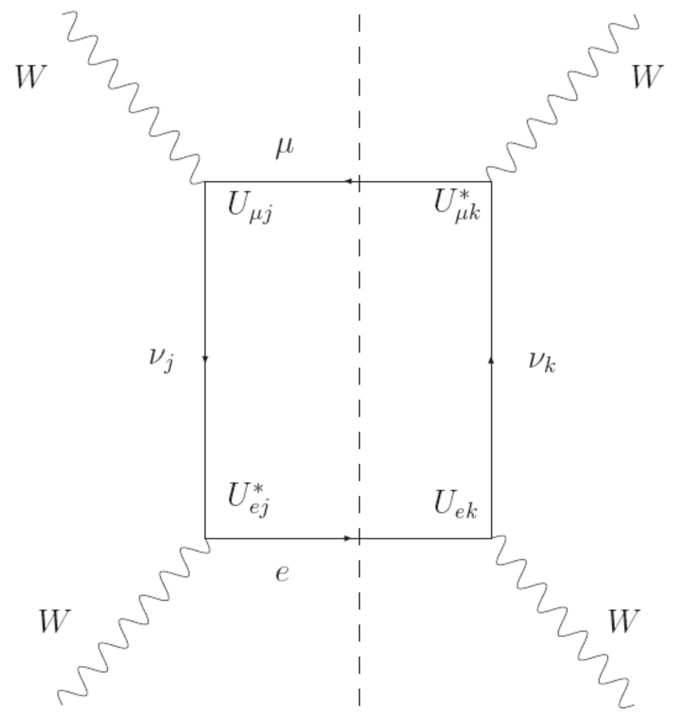
1. Galactic β-beam
2. Source dynamics
   (p-p vs. p-gamma)
   (partial pion-decay chain)
3. ν decay (15 minutes of fame)
4. Vacuum resonance
   (MaVaNs, LIV vector)
5. Pseudo-Dirac ν oscillations

E.g., pure β beam evolves to

$$\underline{\nu}_e \rightarrow \quad 2/3\, \underline{\nu}_1 + 1/3\, \underline{\nu}_2$$

$$\rightarrow \underline{\nu}_e : \underline{\nu}_\mu : \underline{\nu}_\tau \quad = \quad 5:2:2$$

Mixing (phase-averaged oscilllations)



$$P(\nu_\mu \to \nu_e) = \left(\sum_j U_{\mu j} U^*_{ej}\, e^{-i\phi_j}\right) \left(\sum_k U_{\mu k} U^*_{ek}\, e^{-i\phi_k}\right)^* .$$

$$P(\nu_\mu \to \nu_e) \xrightarrow{\text{phase average}} \left(\sum_j U_{\mu j} U^*_{ej}\right) \left(\sum_k U_{\mu k} U^*_{ek}\right)^* \times \delta_{jk} = \sum_j |U_{\mu j}|^2 |U_{ej}|^2 .$$

Let $(\underline{U})_{\alpha j} \equiv |U_{\alpha j}|^2$. Then,

$$P(\nu_\alpha \to \nu_\beta) \xrightarrow{\text{phase average}} (\underline{U})(\underline{U})^T .$$

Neutrino Flavor Ratios for various Astro processes

Process:	Flavors at Source $(\nu_e{:}\nu_\mu{:}\nu_\tau)$	Flavors at Earth $(\nu_e{:}\nu_\mu{:}\nu_\tau)$
Complete $\pi^\pm \to e^\pm \overset{(-)}{\nu_e}\ \nu_\mu \overline{\nu}_\mu$ decay	1:2:0	1:1:1
Incomplete $\pi^\pm \to \mu^\pm \overset{(-)}{\nu_\mu}$	0:1:0	4:7:7
β-beam	pure $\overline{\nu}_e$	5:2:2
Virtual Black Hole Spacetime	Any	1:1:1

plus TriMinimal corrections !

(to be determined)

Summary

Neutrinos from the Cosmos:

First non-atmospheric neutrino event
is "just around the corner"
[like the Higgs and SUSY??]

Next one to ten years will be critical, and,
the deities/gods willing,
most fruitful !

If the statistics are forthcoming,
neutrnio flavor can play a big role.

And the TriMinimal Parametrization keeps it simple.