Anti de Sitter Supersymmetry in N=1 superspace

2008 Phenomenology Symposium
University of Wisconsin-Madison, April 29, 2008

In collaboration with Jonathan Bagger (JHU)

Chi Xiong
Purdue University
2008

References

Gates, Grisaru, Rocek and Siegel, *Superspace or One Thousand and One Lessons in Supersymmetry* 1983.
Wess and Bagger, *Supersymmetry and Supergravity* 1992.
Buchbinder and Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace* 1998.

A. L. Besse, *Einstein Manifolds,1986*.

L.Alvarez-Gaume and D.Z.Freedman, Commun.Math.Phys.80(1981) 443; 91(1983) 87
J. Bagger and E. Witten, Nucl. Phys. B222 (1983) 1
E.Sezgin and Y.Tanii, hep-th/9412163
B.de Wit, B.Kleijn and S.Vandoren. hep-th/9808160
B. de Wit, M. Rocek, S. Vandoren, JHEP 0102 (2001) 039
H. Lu, C.N. Pope and P.K. Towsend, Phys. Lett. B391(1997) 39, hep-th/9607164;
E. Shuster, hep-th/9902129
E.Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, R. Halbersma, S. Vandoren, A. Van Proeyen, JHEP 0210 (2002) 045 and hep-th/0403045

N. Marcus, A.Sagnotti and W. Siegel, Nucl. Phys. B224(1983) 159
B. Milewski, Nucl. Phys. B217(1983) 172-188
C.Hull, A.Karlhede, U.Lindstrom and M.Rocek, Nucl.Phys.B266(1986) 1.

N. Arkani-Hamed, T. Gregoire and J. Wacker, hep-th/0101233
D. Marti and A. Pomarol, hep-th/0106256
A.Hebecker, hep-ph/0112230
B.Gregoire, R. Rattazzi, C.Scrucca, A. Strumia and E. Trincherini, hepth/0411216
J. Bagger and C. Xiong, hep-th/0601165
S.J. Gates, Jr., S. Penati and G. Tartaglino-Mazzucchelli, hep-th/0604042

Motivations

- Study the brane-plus-bulk world scenario
- Find AdS/CFT correspondence
- Simplify supergravity matter couplings

Toy Model

4D scalar-gravity model

$$L = \frac{1}{2}\sqrt{g}\,(\partial^{\mu}\phi\,\partial_{\mu}\phi + \frac{1}{6}\phi^2 R)$$

Local dilatations

$$\delta\phi = \lambda_D \phi, \qquad \delta g_{\mu\nu} = -2\lambda_D g_{\mu\nu}$$

Poincare action

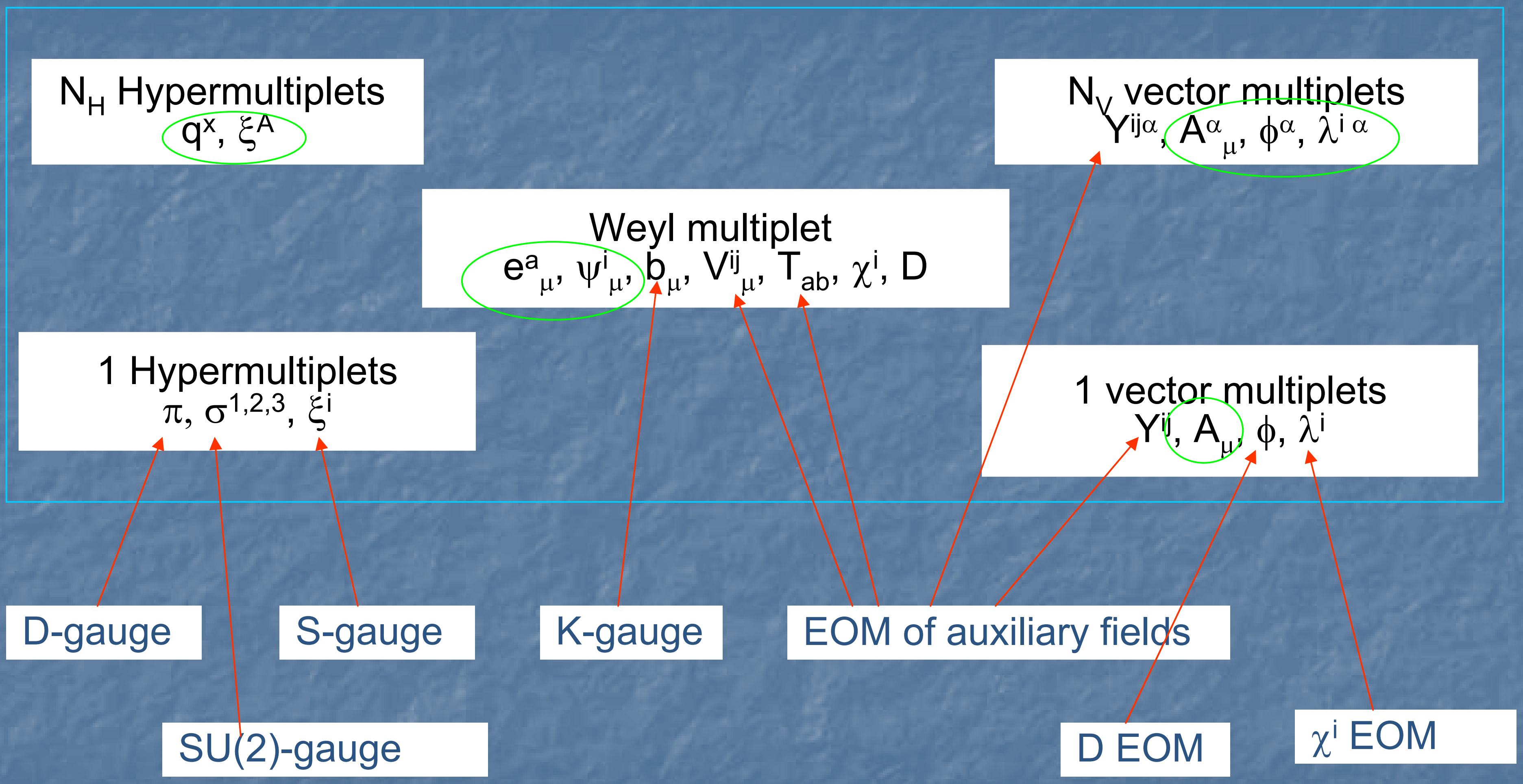
$$\phi^2 = \frac{6}{\kappa^2} \xrightarrow{\text{gauge fixing}} L = \frac{1}{2\kappa^2}\sqrt{g}\,R$$

Supersymmetric generalization

$g_{\mu\nu} \longrightarrow$ Weyl multiplet

$\phi \longrightarrow$ hyper-multiplet/+vector-multiplet

N=2 Conformal supergravity with matter couplings



Superconformal symmetry breaking

Conformal Supergravity $\longrightarrow$ Poincare supergravity

An example of AdS/CFT

A field of mass m on AdS space is related to the dimension of a conformal field on the boundary by (E. Witten, "Anti de Sitter space and holography", hep-th/9802150)

$$m^2 = \Delta(\Delta - d) \quad \Leftrightarrow \quad \Delta = \frac{1}{2}(d + \sqrt{d^2 + 4m^2})$$

In the simplest on-shell Usp(2) supersymmetric massive scalar theory on AdS_5, there are 2 complex scalar fields with masses given by

$$m_{\phi_1}^2 = (c^2 - c - \frac{15}{4})\lambda^2 = (c + \frac{3}{2})(c - \frac{5}{2})\lambda^2$$
$$m_{\phi_2}^2 = (c^2 + c - \frac{15}{4})\lambda^2 = (c + \frac{5}{2})(c - \frac{3}{2})\lambda^2$$

Other examples : Kaluza-Klein harmonics (Kim, Romans and van Nieuwenhuizen, 1985)

Methods and Tools

- Higher dimensional supersymmetric theories have to be the extended ones (N > 1) which receive more constraints
- It is not easy to study extended susy theories in components
- There are “specially designed” tools for extended susy theories -------- harmonic superspace and projective superspace, with an infinite number of auxiliary fields
- We will use N=1 superspace to study extented susy theories

  - N=1 superspace is well-known
  - It is well-suited for the bulk-brane world scenario
  - It works in a very general way

Example I: N=2 Vector Multiplet (abelian)

It has a gauge field A_M, a 4-component Dirac gaugino $\lambda_{1,2}$, and a scalar Σ, which can be put in superfields V and ϕ as

$$V = -\theta\sigma^m\bar{\theta}A_m + i\bar{\theta}^2\theta\lambda_1 - i\theta^2\bar{\theta}\bar{\lambda}_1 + \frac{1}{2}\bar{\theta}^2\theta^2 D$$

$$\phi = \frac{1}{\sqrt{2}}(\Sigma + iA_5) + \sqrt{2}\theta\lambda_2 + \theta^2 F$$

The 5D gauge transformation

$$V \to V + \Lambda + \bar{\Lambda}$$

$$\phi \to \phi + \partial_5\Lambda$$

Gauge invariant action

$$S_5^A = \int d^5x\, W^\alpha W_\alpha \,|_{\theta^2} + h.c. + (\partial_5 V - \phi - \bar{\phi})^2 \,|_{\theta^2\bar{\theta}^2}$$

Question: Where is the second susy?

Second Supersymmetry

The second supersymmetry transformations

$$\delta_\xi V = 2(\phi + \bar{\phi} - \partial_5 V)(\theta\xi + \bar{\theta}\bar{\xi})$$
$$\delta_\xi \phi = -\xi W$$

Two susy transformations close into 5D translations (off-shell)

Non-abelian theories are more complicated but they work in the same way…

The second supersymmetry transformation is important especially in the N=2 hypermultiplet cases without extra gauge symmetry

Example II: 4D N=2 Nonlinear Sigma Models

Its characteristic geometry is called "hyperKahler" geometry, which has 3 complex structures (I, J, K). We consider a hyperKahlerian manifold as a Kahler manifold with an additional holomorphic sympletic structure.

We start with an N=1 sigma model so the first susy is manifest

$$S_4 = \int d^4x d^4\theta \;\; K(\Phi, \bar{\Phi})$$

Ansatz for the second susy

$$\delta_\eta \Phi^a = \frac{1}{2}\bar{D}^2[N^a(\theta\eta + \bar{\theta}\bar{\eta})]$$
$$\delta_\eta \bar{\Phi}^b = \frac{1}{2}D^2[\bar{N}^{\bar{b}}(\theta\eta + \bar{\theta}\bar{\eta})],$$

We require that the action S_4 be invariant under the second susy and the second susy commutator [$\delta_{\eta\,2}$, $\delta_{\xi\,2}$] closes into the N=2 algebra, without using the equations of motion. (however, the closure of first susy and second susy [$\delta_{\eta\,1}$, $\delta_{\xi\,2}$] need the equations of motion)

$$N^a = g^{a\bar{b}}\bar{N}_{\bar{b}}, \;\; \bar{N}_{\bar{b}} = \bar{N}_{\bar{b}}(\Phi, \bar{\Phi}).$$

We then are able to identify the sympletic 2-forms and found that they are covariantly constants. We also derive some integrability conditions which can characterize the hyperkahlerian geometry, e.g. 3 complex structures, Ricci flatness...

Define $\Omega_{ab} \equiv \nabla_a N_b = \partial_a N_b - \Gamma^c_{ab} N_c$

Conditions for invariance of the action and the closure of the algebra

1. $\Omega_{ab} = -\Omega_{ba}$
2. $\Omega_{ab},\ \bar{\Omega}_{\bar{a}\bar{b}}$ **are covariantly constant**
3. $\Omega^a{}_{\bar{c}}\ \Omega^{\bar{c}}{}_b = -\delta^a{}_b,$

3 complex structures

$$J^1 = \begin{pmatrix} 0 & -i\Omega^a{}_{\bar{b}} \\ i\Omega^{\bar{a}}{}_b & 0 \end{pmatrix} \qquad J^2 = \begin{pmatrix} 0 & \Omega^a{}_{\bar{b}} \\ \Omega^{\bar{a}}{}_b & 0 \end{pmatrix} \qquad J^3 = \begin{pmatrix} -i\delta^a{}_b & 0 \\ 0 & i\delta^{\bar{a}}{}_{\bar{b}} \end{pmatrix}.$$

The second susy depends on K and Ω_{ab}

$$N^a = -K_b \Omega^{ba}, \qquad N_a = K_{\bar{b}} \Omega^{\bar{b}}{}_a.$$

An example of HyperKahlerian Manifolds
Eguchi-Hanson T*CP(1) Model

Hyperkahler potential

$$K(X,Y,\bar{X},\bar{Y}) = \sqrt{1+\rho^4} - Log\frac{1+\sqrt{1+\rho^4}}{\rho^2}, \qquad \rho = \sqrt{X\bar{X}+Y\bar{Y}},$$

HyperKahler metric

$$g_{a\bar{b}} = \frac{1}{\rho^4\sqrt{1+\rho^4}}\begin{pmatrix}\rho^6+Y\bar{Y} & -Y\bar{X}\\ -X\bar{Y} & \rho^6+X\bar{X}\end{pmatrix}$$

Holomorphic 2-form Ω_{ab}

$$\Omega_{ab} = \bar{\Omega}_{\bar{a}\bar{b}} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

The second susy

$$N^X = +\frac{\sqrt{1+\rho^4}}{\rho^2}\bar{Y}, \quad N^Y = -\frac{\sqrt{1+\rho^4}}{\rho^2}\bar{X}.$$

N=2 Superpotential and Tri-holomorphic Killing Vectors

In N=1 we can add arbitrary superpotential. In N=2 the superpotential terms receive constraints which are described by the tri-holomorphic Killing vectors (t.h.k.v), which means they are "holomorphic" with respect to each complex structure.

N=2 superpotential depends on t.h.k.v. $P_a = -i\,\Omega_{ab} X^b$

$$S_4 = \int d^4x d^4\theta \; K(\Phi,\bar{\Phi}) + \int d^2\theta \; s\, P(\Phi) + \int d^2\bar{\theta} \; s^* \, \bar{P}(\bar{\Phi}),$$

The t.h.k.v have following properties:

$$\delta_\eta \Phi^a = \frac{1}{2}\Omega^{ab}\bar{D}^2[K_b(\theta\eta + \bar{\theta}\bar{\eta})] - 2(s+s^*)\Omega^{ab}P_b\theta\eta$$
$$\delta_\eta \bar{\Phi}^a = \frac{1}{2}\bar{\Omega}^{\bar{a}\bar{b}}D^2[K_{\bar{b}}(\theta\eta + \bar{\theta}\bar{\eta})] - 2(s+s^*)\bar{\Omega}^{\bar{a}\bar{b}}\bar{P}_{\bar{b}}\bar{\theta}\bar{\eta},$$

- They generate isometry leaving 3 complex structure invariant
- Their norm is exactly the scalar potential
- 1st and 2nd susy close into t.h.k.v.

Define $X^a \equiv i\,\Omega^{ab} P_b$

Killing equation → 1. $\nabla_a \bar{X}_{\bar{b}} + \nabla_{\bar{b}} X_a = 0$

Tri-holo. condition → 2. $\Omega^a{}_{\bar{c}} \nabla_{\bar{b}} \bar{X}^{\bar{c}} - \Omega^c{}_{\bar{b}} \nabla_c X^a = 0.$

This is what N=2 susy tells us. The t.h.k.v. should be related to the central charge when other commutators are calculated.

5D Nonlinear Sigma Models in N=1 superspace

We start from N=2 4D superfields and consider x^5 as a label, then ask the closure of 2 susy $[\, \delta_{\eta\, 1}, \delta_{\xi\, 2}\,] \propto \partial_5$ so we know how to handle $\partial_5\, \phi$. We found the constraint on how the action depends on $\partial_5\, \phi$ which leads to the correct action in components.

$$S_5 = \int d^5x d^4\theta \; K(\Phi, \bar{\Phi}) + \int d^2\theta \;\; P(\Phi, \partial_5\Phi) \;\; + \;\; h.c.$$

Now $\Phi = \Phi\,(x^\mu, x^5)$ and the "superpotential" $P\,(\Phi^a, \partial_5\, \Phi^b\,)$ gives the kinetic terms of 5th dimensional pieces. The closure and 5D Lorentz invariance lead to the constraint →

$$\frac{\partial P}{\partial \Phi^a} - \frac{\partial}{\partial_5}\frac{\partial P}{\partial \partial_5 \Phi^a} = -\Omega_{ab}\partial_5\Phi^b.$$

Which can be solved by $H_a\,(\,\Phi)$ provided that it satisfies

$$\Omega_{ab} = H_{a,b} - H_{b,a}$$

for $P(\Phi) = H_a\, \partial_5\, \Phi^a$

With proper definition of 5D spinors and Γ matrices, the component action becomes fully Lorentz invariant. Note that Ω tensors are absorbed.

$$\begin{aligned} S_5^{\mathsf{A}} \;\; = \;\; & -g_{a\bar{b}}\partial_M A^a \partial^M A^{b*} - i g_{a\bar{b}} \bar{\Psi}^b \Gamma^M (\partial_M \Psi^a + \Gamma^a_{cd} \partial_M A^d \Psi^c) \\ & -\frac{1}{40} R_{a\bar{b}c\bar{d}} (\bar{\Psi}^b \Gamma_M \Psi^a)(\bar{\Psi}^d \Gamma^M \Psi^c). \end{aligned}$$

5D Warped Gravitational Background

We propose a set of rules as guide to constructing supersymmetric theories in the warped gravitational background, e. g. AdS_5 background, by warping the θ variables. The rules are quite simple:

$$\theta \to \vartheta, \quad \bar\theta \to \bar\vartheta, \quad D_\alpha \to \mathcal{D}_\alpha, \quad \bar D_{\dot\alpha} \to \bar{\mathcal{D}}_{\dot\alpha}, \quad \int d^5x \to \int d^5x\sqrt{-g}$$

For AdS_5 metric

$$ds^2 = e^{-2\lambda z}\eta_{mn}dx^m dx^n + dz^2$$

The warping of θ is

$$\vartheta = e^{-\frac{1}{2}\lambda z}\ \theta, \qquad \bar\vartheta = e^{-\frac{1}{2}\lambda z}\ \bar\theta$$

which leads to

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m \quad \to \mathcal{D}_\alpha \equiv \frac{\partial}{\partial\vartheta^\alpha} + i\sigma^a_{\alpha\dot\alpha}\bar\vartheta^{\dot\alpha}e_a{}^m\partial_m \\ \bar D_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m \to \bar{\mathcal{D}}_{\dot\alpha} \equiv -\frac{\partial}{\partial\bar\vartheta^{\dot\alpha}} - i\vartheta^\alpha\sigma^a_{\alpha\dot\alpha}e_a{}^m\partial_m \end{aligned}$$

$\mathcal{D}_\alpha = L\, D_\alpha - 2\,(D^\beta L)\, M_{\beta\alpha}$,

$L= \phi^{1/2}\, \bar\phi^{-1}$, with conformal compensator (fixed) $\phi=e^{1/2\, \lambda z}$

The N=1 superspace is “warped”!

N=2 Abelian Gauge Theory in AdS_5

For N=2 vector multiplet in 5D, the rules lead to the correct component action. But that is not enough as we need Killing spinors which are usually functions of coordinates. One has to solve Killing spinor equation in AdS_5.

In flat superspace

$$S_5^A = \int d^5x \left[\frac{1}{4g^2} \int d^2\theta \, W^\alpha W_\alpha + \text{h.c.} + \frac{1}{g^2} \int d^4\theta \, e^{-2\lambda z} (\partial_5 V - (\chi + \bar{\chi}))^2 \right]$$

Some component fields are rescaled by the warped factor

$$\begin{aligned} V &= -\theta\sigma^a\bar{\theta}\, \delta_a{}^m A_m + i\bar{\theta}^2\theta e^{-\frac{3}{2}\lambda z}\lambda_1 - i\theta^2\bar{\theta}e^{-\frac{3}{2}\lambda z}\bar{\lambda}_1 + \frac{1}{2}\bar{\theta}^2\theta^2 e^{-2\lambda z} D \\ \chi &= \frac{1}{\sqrt{2}}(\Sigma + iA_5) + \sqrt{2}\theta e^{-\frac{1}{2}\lambda z}\lambda_2 + \theta^2 e^{-\lambda z} F \end{aligned}$$

The second Q-susy has a 4D conformal S-susy piece in ξ^A.

$$\begin{aligned} \delta_{\eta_2} V &= 2(\chi + \bar{\chi} - \partial_5 V)(\theta\, \eta_2^o + \bar{\theta}\, \bar{\eta}_2^o) - \lambda\, \xi^A D_A V \\ \delta_{\eta_2}\chi &= -e^{2\lambda z}\, \eta_2^o\, W - \lambda\, \xi^A D_A \chi \end{aligned}$$

From the expression of ξ_α, we see its explicit dependence of x^m.

$$\begin{aligned} \xi^a &= -2(\theta\sigma^a\bar{\sigma}^b\eta_2 + \bar{\eta}_2\bar{\sigma}^b\sigma^a\bar{\theta})x^m\hat{e}_{mb} + 2i(\bar{\theta}^2\theta\sigma^a\bar{\eta}_2 - \theta^2\eta_2\sigma^a\bar{\theta}) \\ \xi_\alpha &= 2\theta_\alpha(\bar{\theta}\bar{\eta}_2^o) + ix^m\delta_m{}^a(\sigma_a\bar{\eta}_2^o)_\alpha + 2\eta_{2\alpha}^o\theta^2 \end{aligned}$$

The Killing spinor equation in AdS_5 can be found from the susy transformation law of gavitini, which is $D_M \Psi_i + \lambda /2\ \Gamma_M (q \cdot \sigma)_i^{\ j} \Psi_j = 0$. Its solutions can be expressed in the combination of two Weyl spinors (η^o_1 and η^o_2 are independent constant Weyl spinors)

η^o_1 and η^o_2 represent the first susy and the second susy respectively. The first susy is generated by the operator

$$\eta_1 Q + \bar{\eta}_1 \bar{Q} = \eta_1^o Q + \bar{\eta}_1^o \bar{Q}$$

$$\psi_1^- = e^{\frac{1}{2}\lambda z}\eta_2^o, \quad \psi_2^+ = e^{-\frac{1}{2}\lambda z}(\eta_1^o + i\lambda x^m \delta_m^{\ a} \sigma_a \bar{\eta}_2^o)$$

In the left diagram it is shown how the Killing spinor splits into two supersymmetries.

$$\delta_{\eta_2} V = 2(\chi + \bar{\chi} - \partial_5 V)(\vartheta\ e^{\frac{1}{2}\lambda z}\ \eta_2^o + \bar{\vartheta}\ e^{\frac{1}{2}\lambda z}\ \bar{\eta}_2^o) - \lambda\ \zeta^A \mathcal{D}_A V$$

The 5D Killing vectors can be obtained by lifting 4D conformal Killing vectors into 5D.

$$\xi^a(x,z) = \hat{\xi}^a(x) - \frac{e^{2\lambda z}}{2\lambda^2}\hat{\partial}^a \sigma(\hat{\xi})$$

$$\xi^5(x,z) = \frac{1}{\lambda}\sigma(\hat{\xi})$$

$$\sigma(\hat{\xi}) = \lambda_D - 2(x \cdot f).$$

N=2 Hypermultiplet in AdS$_5$

This approach also works for the hypermultiplets

$$S_{5D} = \int d^5x[\int d^4\theta e^{-2\lambda z} K(\Phi, \bar{\Phi}) + \int d^2\theta e^{-3\lambda z} H_a \partial_5 \Phi^a + h.c.]$$

2nd susy

$$\delta_\eta \Phi^a = \frac{1}{2} e^{\lambda z} \bar{D}^2[\Omega^{ab} K_b(\theta\eta + \bar{\theta}\bar{\eta})] - 12\lambda\Omega^{ab} H_b \theta\eta - \lambda\epsilon_s^A D_A \Phi^a$$

Constraints:

$$K_a \Omega^{ab} H_b = K_{\bar{a}} \bar{\Omega}^{\bar{a}\bar{b}} \bar{H}_{\bar{b}}$$

$$i\Omega_{ac} \nabla_b X^c - i\Omega_{bc} \nabla_a X^c = \Omega_{ab}$$

where $X^a = i\,\Omega^{ab} H_b$ and $Y^a = i\,X^a$ is the homothetic Killing vector

Then one can use gauge fixing conditions to find more general HyperKahler potentials in the AdS background.

$$V_{scalar} = -\sqrt{g}\,\lambda^2(-9Y^aY_a - 12K_aY^a)$$

The scalar potential depends on the hyperKahler potential and the corresponding homothetic Killing vector

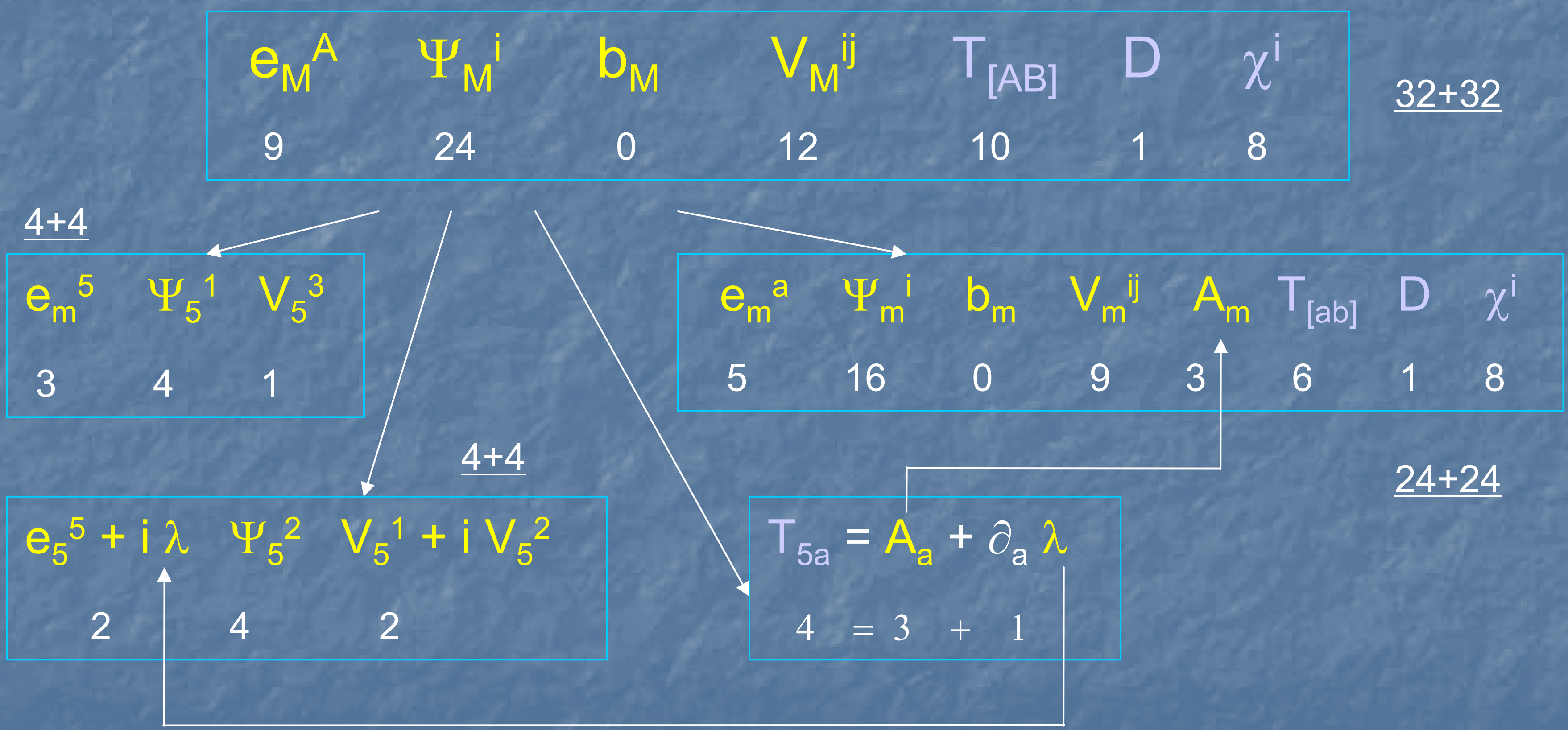
$$K(\phi,\bar{\phi}) = \bar{\phi}^1\phi^1 + \bar{\phi}^2\phi^2$$

$$\longrightarrow \quad m^2_{\phi_1} = (c^2 - c - \frac{15}{4})\lambda^2, \quad m^2_{\phi_2} = (c^2 + c - \frac{15}{4})\lambda^2$$

This is what we expected from the AdS/CFT correspondence.

Decomposition of 5D Weyl Multiplet

This diagram shows how to break a 5D N=2 Weyl multiplet into a 4D N=2 Weyl multiplet plus one chiral superfield and one vector superfield, naively. In reality they will mix with another N=2 vector multiplet to give the physical radion and graviphoton plus their super partners. Hence the problem reduces to the 4D N=2 compensators.



Summary

- The N=1 superspace formulation can be used to build supersymmetric models in higher dimensional spacetime, by constructing N=2 hypermultiplets, vector multiplets and tensor multiplets etc.
- Using conformal compensators, the matter couplings of supergravity could be simplified.
- We generalize this approach in two directions: One is to have larger symmetry (from Poincare group to conformal group); another is to include gravitational background like the AdS spaces. This tool may help to understand the correspondence between AdS gravity and conformal field or string theories.