# Naturalness, Neutrinos, and GUTs

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# SO(10) Yukawas and Seesaw

Break SO(10) to SM via: $\langle \mathbf{16}_H \rangle, \langle \mathbf{45}_H \rangle \sim M_{GUT}$ ; $\langle \mathbf{10}_H \rangle, \langle \mathbf{16}'_H \rangle \sim M_{WEAK}$.

Realistic Yukawa couplings are built from several operators.
$\mathbf{16}_i \mathbf{16}_j \mathbf{10}_H$: Contributes equally to all Yukawa couplings. (symm.)
$\mathbf{16}_i \mathbf{16}_j \mathbf{10}_H \mathbf{45}_H$: Differentiates quarks and leptons. (anti-symm.)
$\mathbf{16}_i \mathbf{16}_j \mathbf{16}_H \mathbf{16}'_H$: Contributes only to $Y_d$ and $Y_e$. (symm. or antisymm.)

**Neutrino terms**
$\nu^i M_D^{ij} N^j$ recieves contributions from $\mathbf{16}_i \mathbf{16}_j \mathbf{10}_H, \mathbf{16}_i \mathbf{16}_j \mathbf{10}_H \mathbf{45}_H$.
$\frac{1}{m}(M_R)_{ij}\, \mathbf{16}_i \mathbf{16}_j \overline{\mathbf{16}}_H \overline{\mathbf{16}}_H \rightarrow N^i M_R^{ij} N^j$.
Integrating out the heavy fields leaves

$$M_\nu \simeq -M_D M_R^{-1} (M_D)^T \sim \frac{M_{WEAK}^2 m}{M_{GUT}^2} \sim 0.1\text{eV}.$$

# Philosophy

- This set of operators is minimal for a realistic model, but more than sufficient to numerically satisfy experimental values.
- A number of economical models exist in the literature. Family symmetries used to restrict the form of the Yukawa operators, leading to good fits for charged fermion sector. (e.g. Albright and Barr, 2000; Babu, Pati and Wilczek, 2000; Raby and Dermisek, 2000)
- Neutrinos more problematic: generically additional degrees of freedom, but models often require some fine tuning.
- We approach the problem without assuming a priori family symmetries and impose naturalness, i.e., no cancellations between two terms to a lower order of magnitude. Analyze what constraints experiment imposes on Yukawa structure.
- Focus on orders of magnitude only.

# Parameterization

Any matrix may be decomposed as

$$M \equiv L D R^\dagger$$

where

$$L^\dagger M M^\dagger L = R^\dagger M^\dagger M R = D^2 \equiv \mathrm{diag}\left(\eta^2, \epsilon^2, 1\right),$$

Hierarchical eigenvalues $\eta \ll \epsilon \ll 1$ will be naturally generated if

$$D \equiv \mathrm{diag}\left(\eta, \epsilon, 1\right), \quad L \sim \begin{pmatrix} 1 & \mu' \sqrt{\frac{\eta}{\epsilon}} & \nu' \sqrt{\eta} \\ \mu' \sqrt{\frac{\eta}{\epsilon}} & 1 & \rho' \sqrt{\epsilon} \\ \nu' \sqrt{\eta} & \rho' \sqrt{\epsilon} & 1 \end{pmatrix}, \quad R \sim \begin{pmatrix} 1 & \mu \sqrt{\frac{\eta}{\epsilon}} & \nu \sqrt{\eta} \\ \mu \sqrt{\frac{\eta}{\epsilon}} & 1 & \rho \sqrt{\epsilon} \\ \nu \sqrt{\eta} & \rho \sqrt{\epsilon} & 1 \end{pmatrix}.$$

Based on the quark hierarchy $\eta \sim 10^{-(4-5)}$ and $\epsilon \sim 10^{-(2-3)}$.
$\mu, \nu, \rho \lesssim 1$ for naturalness.

# Geometric Hierarchy

For $\mu, \nu, \rho \sim 1$

$$M \sim \begin{pmatrix} \leq \eta & \sqrt{\eta\epsilon} & \sqrt{\eta} \\ \sqrt{\eta\epsilon} & \leq \epsilon & \sqrt{\epsilon} \\ \sqrt{\eta} & \sqrt{\epsilon} & 1 \end{pmatrix}.$$

- Order symmetric: Sum of symmetric and antisymmetric matrices shouldn't cancel: $M_{ij} \sim M_{ji}$.
- Off diagonal entries are dominant or codominant in determining smaller eigenvalues.
- Easily generated via $U(1)$ symmetry using Froggatt-Nielsen mechanism. (Froggatt-Nielsen, 1979)
- Default natural structure given strong quark mass hierarchy. (Least restricted.)

# Experimental Neutrino Data

Neutrinos are a critical part of the impetus for, and therefore construction of SO(10) models. In recent years a consensus picture of the data has emerged, including the Large Mixing Angle (LMA) solution to the solar neutrino puzzle. Absolute masses are unknown but oscillation experiments require:

$$\tan^2\theta_{12} = 0.45 \pm 0.05\ ; \qquad \Delta m^2_{\text{sol}} \simeq (8.0 \pm 0.3) \times 10^{-5}\ \text{eV}^2\ ;$$

$$\sin^2 2\theta_{23} = 1.02 \pm 0.04\ ; \qquad \Delta m^2_{\text{atm}} \simeq (2.5 \pm 0.2) \times 10^{-3}\ \text{eV}^2\ ;$$

$$\sin^2 2\theta_{13} = 0 \pm 0.05.$$

(Strumia and Vissani, hep-ph/0606054)

Cosmology, $0\nu\beta\beta \to \Sigma\, m_\nu \lesssim 1$ eV.

# Tri-bimaximal Solution

Set mixing angles $\theta_{12} = 35.2^\circ, \theta_{13} = 0^\circ, \theta_{23} = 45^\circ$:

$$V_{\text{PMNS}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

Compare to CKM matrix in quark sector (Yao et al., 2006[PDG])

$$V_{\text{CKM}} = \begin{pmatrix} 1 & 0.226 \pm 0.002 & [4.3 \pm 0.3] \times 10^{-3} \\ 0.23 \pm 0.01 & 1 & [4.2 \pm 0.06] \times 10^{-2} \\ [7.4 \pm 0.8] \times 10^{-3} & 3.5 \times 10^{-2} & 1 \end{pmatrix}$$

Strong quark mass hierarchies: $m_u \sim 3.5 \times 10^{-3} m_c \sim 10^{-5} m_t$.

# The Experimental Effective Neutrino Matrix

$V_{\text{PMNS}} \equiv L_e^\dagger L_\nu$

For $M_e$ hierarchical, $L_e \sim \mathbf{1}$, $L_\nu$ dominates.

$M_\nu$ is diagonalized by $L_\nu = R_\nu = V_{\text{PMNS}}$ if

$$M_\nu = V_{\text{PMNS}} D_\nu V_{\text{PMNS}}^\dagger \propto \begin{pmatrix} \left(m_1 + \frac{1}{2} m_2\right) & -\frac{1}{2}\left(m_1 - m_2\right) & \frac{1}{2}\left(m_1 - m_2\right) \\ -\frac{1}{2}\left(m_1 - m_2\right) & \frac{1}{2}\left(\frac{1}{2} m_1 + m_2 + \frac{3}{2} m_3\right) & -\frac{1}{2}\left(\frac{1}{2} m_1 + m_2 - \frac{3}{2} m_3\right) \\ \frac{1}{2}\left(m_1 - m_2\right) & -\frac{1}{2}\left(\frac{1}{2} m_1 + m_2 - \frac{3}{2} m_3\right) & \frac{1}{2}\left(\frac{1}{2} m_1 + m_2 + \frac{3}{2} m_3\right) \end{pmatrix}.$$

We can write the physical masses $m$ in terms of $m_1$:

$$m_1 = e^{i\phi_1} \left|m_1\right|, \quad m_2 = e^{i\phi_2} \sqrt{|m_1|^2 + \Delta_{\text{sol}}^2}, \quad m_3 = \sqrt{|m_1|^2 + \Delta_{\text{sol}}^2 \pm \Delta_{\text{atm}}^2}.$$

Note $+(-)$ corresponds to normal(inverted) hierarchy.

# Effective Matrix Textures

Scanning throught the allowed ranges of $m_1$, we find the following approximate textures.

1. $M_\nu \sim \begin{pmatrix} \lambda & \lambda & \lambda \\ \lambda & 1 & 1 \\ \lambda & 1 & 1 \end{pmatrix}$, corresponding to $m_1 \ll m_2 \simeq \Delta_{\text{sol}}$, normal hierarchy.
2. $M_\nu \sim \begin{pmatrix} 0 & \lambda & \lambda \\ \lambda & 1 & 1 \\ \lambda & 1 & 1 \end{pmatrix}$, corresponding to $2m_1 \simeq m_2 \simeq \frac{2}{\sqrt{3}}\Delta_{\text{sol}}$, $\phi_2 - \phi_1 = \pi$, normal hierarchy.
3. $M_\nu \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, corresponding to $\Delta_{\text{sol}}(\Delta_{\text{atm}}) \lesssim m_1 \simeq m_2 \lesssim \Delta_{\text{atm}}(\sqrt{2}\Delta_{\text{atm}})$, $\phi_2 - \phi_1 = 0$, normal (inverted) hierarchy.
4. $M_\nu \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, corresponding to degenerate masses, $\phi_2 = 0$, $\phi_1 = 0$.
5. $M_\nu \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, corresponding to degenerate masses, $\phi_2 = \pi$, $\phi_1 = \pi$.
6. $M_\nu \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, corresponding to degenerate masses, $\phi_2 - \phi_1 = \pi$.

$\lambda \equiv \frac{\Delta_{\text{sol}}}{\Delta_{\text{atm}}} \simeq 0.2$

# Is there a natural case?

1. $M_\nu \sim \begin{pmatrix} \lambda & \lambda & \lambda \\ \lambda & 1 & 1 \\ \lambda & 1 & 1 \end{pmatrix}$, Violates geometric hierarchy. Generically 1 large angle and eigenvalues $\lambda, 1, 1$.
2. $M_\nu \sim \begin{pmatrix} 0 & \lambda & \lambda \\ \lambda & 1 & 1 \\ \lambda & 1 & 1 \end{pmatrix}$, Violates geometric hierarchy. Generically 1 large angle and eigenvalues $\lambda^2, 1, 1$.
3. $M_\nu \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, Violates geometric hierarchy. Generically 1 large angle and eigenvalues $1, 1, 1$.
4. $M_\nu \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, Degenerate eigenvalues but generically no large mixing angles.
5. $M_\nu \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, Degenerate eigenvalues but generically no large mixing angles.
6. $M_\nu \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, Naturally degenerate masses and large mixing angles ($\theta_{13}$ could be problematic.)

# Deciphering the Seesaw Formula

$$M_\nu \simeq -M_D M_R^{-1} (M_D)^T \to R_D^\dagger M_R^{-1} R_D^* = D_D^{-1} L_D^\dagger M_\nu L_D^* D_D^{-1}.$$

Apply to Case 6 (democratic $M_\nu$ and $M_D$ parameterization.

$$R_D^\dagger M_R^{-1} R_D^* \sim \begin{pmatrix} \frac{1}{\eta^2} & \frac{1}{\eta\epsilon} & \frac{1}{\eta} \\ \frac{1}{\eta\epsilon} & \frac{1}{\epsilon^2} & \frac{1}{\epsilon} \\ \frac{1}{\eta} & \frac{1}{\epsilon} & 1 \end{pmatrix}$$

For $\mu'$, $\nu'$, $\rho' \leq 1$, the $L_D$ rotations are unimportant.

$M_R$ **should exhibit a double hierarchy: eigenvalues scale as** $\eta^2, \epsilon^2, 1$**!**

# Constraints on $R_D$

Imposing naturalness (non-cancellation) on all entries:

$$M_R^{-1} \sim D_D^{-1} L_D^\dagger M_\nu L_D^* D_D^{-1}.$$

Which in turn implies

$$\mu \lesssim \sqrt{\frac{\eta}{\epsilon}}, \quad \nu \lesssim \sqrt{\eta}, \quad \rho \lesssim \sqrt{\epsilon}.$$

For $\eta \sim 10^{-4}$ and $\epsilon \sim 10^{-2}$, this corresponds to $\mu,\ \rho \lesssim 10^{-1}$ and $\nu \lesssim 10^{-2}$. Applying the same conditions to $\mu'$, $\nu'$, and $\rho'$ implies a **Cascade Hierarchy**:

$$M_D \sim \begin{pmatrix} \eta & \eta & \eta \\ \eta & \epsilon & \epsilon \\ \eta & \epsilon & 1 \end{pmatrix}.$$

# Recap

- For any hierarchical texture of $R_D^\dagger M_R^{-1} R_D^*$, $M_R^{-1}$ retains the same or stronger hierarchy.
- $R_D$ smears out $M_R^{-1}$, large entries rotated into smaller.
- Conversely, for a strong hierarchy in $M_R^{-1}$, $R_D$ must be close to diagonal $\rightarrow$constraints on $\mu$, $\nu$ and $\rho$.
- Non-trivial constraints on form of theoretical inputs $M_D$ and $M_R$.

# Implementing a Cascade Hierarchy

Froggat-Nielsen mechanism:
Impose a $U(1) \times \mathbb{Z}_2 \times \mathbb{Z}_2'$ symmetry.

| Field | $\mathbf{16}_1$ | $\mathbf{16}_2$ | $\mathbf{16}_3$ | $\mathbf{10}_H$ | $\phi_1$ | $\phi_2$ | $\phi_3$ |
|---|---|---|---|---|---|---|---|
| U(1) | 2 | 1 | 0 | 0 | $-1$ | 0 | 0 |
| $\mathbb{Z}_2$ | – | – | + | + | + | – | + |
| $\mathbb{Z}_2'$ | – | + | + | + | + | + | – |

$\Phi^{ij}\mathbf{16}_i\mathbf{16}_j\mathbf{10}_H \to M_{10}^{ij}\mathbf{16}_i\mathbf{16}_j\mathbf{10}_H$

$$\Phi = \begin{pmatrix} \frac{1}{m^4}(\phi_1)^4 & \frac{1}{m^4}(\phi_1)^3\phi_3 & \frac{1}{m^4}(\phi_1)^2\phi_2\,\phi_3 \\ \frac{1}{m^4}(\phi_1)^3\phi_3 & \frac{1}{m^2}(\phi_1)^2 & \frac{1}{m^2}\phi_1\,\phi_2 \\ \frac{1}{m^4}(\phi_1)^2\phi_2\,\phi_3 & \frac{1}{m^2}\phi_1\,\phi_2 & 1 \end{pmatrix}.$$

Thus

$$M_{10} \sim \begin{pmatrix} \zeta^4 & \zeta^4 & \zeta^4 \\ \zeta^4 & \zeta^2 & \zeta^2 \\ \zeta^4 & \zeta^2 & 1 \end{pmatrix}.$$

$\zeta \equiv \langle \phi \rangle / m$

# Efficacy of Cascade Hierarchies

Cascade-like hierarchies correspond to small off-diagonal elements/unitary rotations.

- Cascade hierarchies lead to better predictions for 3rd generation CKM couplings.
- But, relatively large Cabibbo angle is more consistent with geometric hierarchy.
- Small couplings in $\mathbf{16}_i\mathbf{16}_j\mathbf{10}_H\mathbf{45}_H$ lead to difficulties in naturally fitting $M_u$, $M_d$, $M_e$ hierarchies.

Problems may point to mixed cascade/geometric hierarchies.
Can we avoid these conclusions by relaxing any assumptions?

# Lopsidedness

At this point we seem to have somewhat robust conclusions and some potential difficulties in building a complete theory. Can we modify our assumptions to come to different conclusions?

It is possible to build order asymmetric Yukawa matrices $\rightarrow$ Large off-diagonal mixing terms.

| Hierarchy | $M$ | $L$ | $R$ |
|---|---|---|---|
| Geometric | $\begin{pmatrix} \epsilon & \sqrt{\epsilon} \\ \sqrt{\epsilon} & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \sqrt{\epsilon} \\ \sqrt{\epsilon} & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \sqrt{\epsilon} \\ \sqrt{\epsilon} & 1 \end{pmatrix}$ |
| Cascade | $\begin{pmatrix} \epsilon & \epsilon \\ \epsilon & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}$ |
| Lopsided | $\begin{pmatrix} \epsilon & \epsilon \\ 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ |

Note trade-off in Lopsided matrices: as $R$ becomes more mixed, $L$ becomes closer to diagonal.

# Large Rotations

Lopsided matrices constructed so as to generate large $R_d$ and $L_e$.
$L_d$ (hence $R_e$) must remain small for consistency with CKM.
$L_e$ no longer negligible in $V_{\text{PMNS}}$.

$$R_D^\dagger M_R^{-1} R_D^* = D_D^{-1} V_0 M_\nu' V_0^T D_D^{-1},$$

where

$$M_\nu' \equiv V_{\text{PMNS}} D_\nu V_{\text{PMNS}}^T = L_e^\dagger M_\nu L_e^*, \qquad V_0 \equiv L_D^\dagger L_e \ .$$

$M_\nu'$ has same experimentally allowed forms as $M_\nu$ previously. Same analysis applies except:

- $L_e$ accounts for some (all?) large angles in $V_{\text{PMNS}}$.
- $V_0 M_\nu' V_0^T = L_D^\dagger M_\nu L_D$ no longer approximately democratic.
- $R_D^\dagger M_R^{-1} R_D^*$ has a weaker hierarchy (not doubled).
- Cascade-like constraints on $M_D$ are weakened.

# Outlook

- SO(10) GUTs make a compelling case for gauge coupling and fermion representation unification.
- Additionally, they provide the most appealing mechanism for explaining small, non-zero $m_\nu$.
- The appearance of large mixing angles and a weakly hierarchical or degenerate mass spectrum is inconsistent with simple geometric Yukawa coupling patterns.
- For order symmetric matrices, these facts point to a strong hierarchy in $M_R$ and cascade-like Yukawa couplings.
- These features can be built into models with simple symmetries, however simple implementations run into difficulties fitting quark sector.
- Lopsided model offer a way to mitigate these conclusions, a successful model may incorporate features of lopsided and cascade textures.
- Experimental determination of $\theta_{13}$, overall mass scale remain crucial to discriminate between potential models.